1-1

$M_e = 9.109 \times 10^{-31}$ Kg

$M_p = 1.673 \times 10^{-27}$ Kg

$G = 6.673 \times 10^{-11} \frac{M^3}{Kg\, s^2}$

$r = .5\ \mathring{A} = 0.5 \times 10^{-10}$ m

$$F_g = \frac{G m_1 m_2}{r^2}$$

$$= \frac{(6.673\times10^{-11})(9.109\times10^{-31})(1.673\times10^{-27})}{(0.5\times10^{-10})^2}$$

$$F_g = 4.068 \times 10^{-47}\ N$$

$$F_e = \frac{K e_1 e_2}{r^2}$$

$K = 8.987 \times 10^9 \frac{Nm^2}{C^2}$

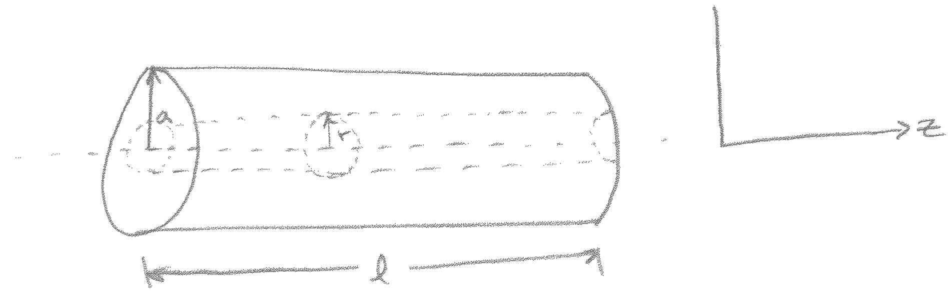
$e = 1.602 \times 10^{-19}$ C

$$F_e = \frac{(8.987\times10^9)(1.602\times10^{-19})^2}{(0.5\times10^{-10})^2}$$

$$F_e = 9.133 \times 10^{-8}\ N$$

$$\frac{F_e}{F_g} = 2.245 \times 10^{39}$$

1-3



$\Delta p \sim l$

$\frac{F_z}{A_v} = \eta \frac{dv_z}{dr}$ $\qquad A_v = 2\pi r l$ {surface area of cylinder}

$\Delta p(\pi r^2) + F_z = 0$ {steady flow}

$\Delta p(\pi r^2) + 2\pi r l \eta \frac{dv_z}{dr} = 0$

$$\frac{dv_z}{dr} = \frac{r\Delta p}{2\eta l}$$

$$v_z - v_0 = \frac{\Delta p}{2\eta l}\int_0^r r' dr'$$

$$v_z = \frac{\Delta p}{4\eta l} r^2 + v_0$$

$\Phi = \int v_z \, dA$ $\qquad A = \pi r^2 \rightarrow dA = 2\pi r \, dr$

$$= \int v_z (2\pi r \, dr)$$

$$= \frac{2\pi \cdot \Delta p}{4\eta l}\int_0^a r^3 dr$$ {ignoring v_0}

$$= \frac{\pi \Delta p}{2\eta l} \frac{r^4}{4}\Big|_0^a$$

$$= \frac{\pi \Delta p}{8\eta l} a^4$$

$$\therefore \Delta p \sim \frac{\Phi \eta l}{a^4}$$

1-5

a) v_0 = motorist speed

τ = reaction time

a = braking deceleration

$v_f^2 = v_0^2 + 2a(\Delta x)$

$0 = v_0^2 - 2a(X_b)$ $\quad X_b$ = braking distance

$X_b = \frac{v_0^2}{2a}$

$X_r = v_0 \tau$

$S_{min} = X_r + X_b$

$\boxed{S_{min} = v_0 \tau + \frac{v_0^2}{2a}}$

b) t = time of amber light

when traveling at v_0 the maximum distance from the intersection when the light turns amber such that the car continues into the intersection with speed v_0 without running the red light is simply

$S_{max} = v_0 t$

c) Given $v_0 > v_{0max} = 2a(t-\tau)$

at v_{0max}

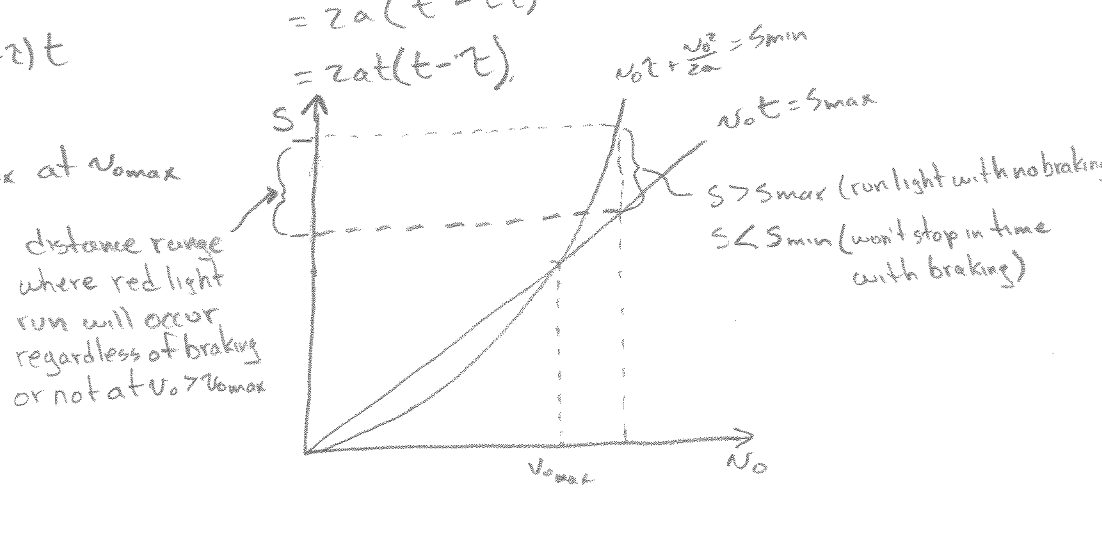
$S_{min} = 2a(t-\tau)\tau + \frac{(2a(t-\tau))^2}{2a}$

$S_{min} = 2a(t-\tau)\tau + 2a(t-\tau)^2 = 2a(t\tau - \tau^2 + t^2 - 2t\tau + \tau^2)$

$= 2a(t^2 - t\tau)$

$S_{max} = 2a(t-\tau)t \qquad = 2at(t-\tau)$

Hence $S_{min} = S_{max}$ at v_{0max}



$V_0 > V_{0max}$ the range in question is

$$S_{max} < S < S_{min}$$
$$v_0 t < s < v_0 \tau + \frac{v_0^2}{2a}$$

d) $\tau = 0.5s$

$t = 5s$

$a = 5\, m/s^2$

$v_{0max} = 2a(t-\tau)$

$= 2(5)(5-.5)$

$= 45\, m/s \cdot \frac{1in}{.0254m} \cdot \frac{1ft}{12in} \cdot \frac{1mile}{5,280ft} \cdot \frac{3600s}{1hr}$

$\boxed{v_{0max} = 101\, mph}$

$v_0 = \frac{2}{3} v_{0max}$

$v_0 = 30\, m/s = 67\, mph$

$S_{min} = v_0 \tau + \frac{v_0^2}{2a}$

$= 30(.5) + \frac{(30)^2}{14}$

$= 79\, m$

$\boxed{S_{min} = 260\, ft}$

$S_{max} = v_0 t$

$= (30)(5)$

$= 150\, m$

$\boxed{S_{max} = 495\, ft}$

1-11

$r = 9.3\times10^7\text{ miles}\cdot\frac{5{,}280\text{ ft}}{1\text{ mile}}\cdot\frac{12\text{ in}}{1\text{ ft}}\cdot\frac{.0254\text{ m}}{1\text{ in.}} = 1.50\times10^{11}\text{ m}$

$T = 1\text{ year}\cdot\frac{365\text{ days}}{1\text{ year}}\cdot\frac{24\text{ hrs}}{1\text{ day}}\cdot\frac{3600\text{ sec}}{1\text{ hr}} = 3.154\times10^7\text{ sec}$

$F_g = \frac{G m_1 m_2}{r^2}$ $m_1 = m_e$

$m_2 = m_s$

$F_{cent} = m_1\frac{V_r^2}{r}$

$F_{cent.} = F_g$

$\frac{m_1 V_r^2}{r} = \frac{G m_1 m_2}{r^2}$

$\frac{V_r^2 r}{G} = m_2$ $V_r = r\omega$, $\omega = \frac{2\pi}{T}$

$m_s = \frac{\omega^2 r^3}{G}$ $= 1.99\times10^{-7}\text{ sec}^{-1}$

$= \frac{(1.99\times10^{-7}\text{ s}^{-1})^2(1.50\times10^{11}\text{ m})^3}{6.67\times10^{-11}\frac{\text{m}^3}{\text{kg}\cdot\text{s}^2}}$

$= 2.00\times10^{30}\text{ kg}\cdot\frac{2.2\text{ lb}}{\text{kg}}$

$= 4.4\times10^{30}\text{ lbs}\times\frac{1\text{ ton}}{2000\text{ lb}}$

$\boxed{m_s = 2.2\times10^{27}\text{ tons}}$

1-13

$r_1 = 25{,}000 \text{ light years} \cdot 2.99\times10^8 \frac{m}{s} \cdot \frac{3600 s}{hr} \cdot \frac{24 hr}{day} \cdot \frac{365 day}{1 year} = 2.357\times10^{20} m$

$V_{T_s} = 175 \frac{miles}{sec} \cdot \frac{5{,}280 ft}{1 mile} \cdot \frac{12 in}{1 ft} \cdot \frac{.0254 m}{in} = 2.82\times10^5 m/s$

$$F_g = \frac{m_s m_G G}{r_1^2}$$

$$F_{cent} = \frac{m V_T^2}{r}$$

$$= \frac{m_s V_{T_s}^2}{r_1}$$

$$F_g = F_{cent}$$

$$\frac{m_s V_{T_s}^2}{r_1} = G\frac{m_s m_G}{r_1^2}$$

$$m_G = \frac{V_{T_s}^2 r_1}{G}$$

For the Sun-Earth system

$$F_g = \frac{G m_e m_s}{r_2^2}$$

$$F_{cent} = \frac{m_e V_{Te}^2}{r_2}$$

$$F_g = F_{cent}$$

$$\frac{G m_e m_s}{r_2^2} = \frac{m_e V_{Te}^2}{r_2}$$

$$G = \frac{V_{Te}^2 r_2}{m_s} \qquad V_{Te} = \omega_e r_2 \qquad \omega = \frac{2\pi}{T} = \frac{2\pi}{1 year} = 1.99\times10^{-7} s^{-1}$$

$$r_2 = 1.50\times10^{11} m$$

$$m_G = \frac{V_{T_s}^2 r_1}{G}$$

$$= V_{T_s}^2 r_1 \cdot \frac{m_s}{V_{Te}^2 r_2}$$

$$= V_{T_s}^2 r_1 \cdot \frac{m_s}{\omega_e^2 r_2^3}$$

$$\boxed{\frac{m_G}{m_s} = \frac{V_{T_s}^2 r_1}{\omega_e^2 r_2^3} = \frac{(2.82\times10^5)^2 (2.357\times10^{20})}{(1.99\times10^{-7})^2 (1.50\times10^{11})^3} = 1.40\times10^{11}}$$

2-1

a) $F = 3000\,lb\text{-}wt$

Convert to MKS units

$1\,lb\ force = 4.448\,N$

$F = mg$

$F = \left(\frac{4.448N}{lb} \cdot 3000lb\right)$

$F = 13344\,N$

$V_1 = 20\,mph = \frac{20\,miles}{1hr} \cdot \frac{5{,}280ft}{1mile} \cdot \frac{12in}{1ft} \cdot \frac{.0254m}{in} \cdot \frac{1hr}{3600sec} = 8.9\frac{m}{s}$

$FV_1 = (13344N)(8.9\frac{m}{s})$

$= 118761\,\frac{N \cdot m}{s}$

$= 118761\,W \cdot \frac{1hp}{746W}$

$\boxed{FV_1 = 160\,hp}$

$V_2 = 160mph \cdot \frac{8.9m/s}{20mph} =$

$= 5V_1$

$FV_2 = 5FV_1$

$\boxed{FV_2 = 800\,hp}$

$V_3 = 300mph \cdot \frac{8.9m/s}{20mph}$

$= 15V_1$

$FV_3 = 15FV_1$

$\boxed{FV_3 = 2400hp}$

b) $P = 500 \text{hp}$

$V_1 = 20 \text{mph} = 8.9 \frac{m}{s}$

$P = 500 \text{hp} \cdot \frac{746 W}{\text{hp}}$

$= 373000 W$

$P = Fv$

$F = \frac{P}{V}$

$F_1 = \frac{P}{V_1}$

$= \frac{373000 W}{8.9 \frac{m}{s}}$

$= 41910.11 N$

$= 41910.11 N \cdot \frac{\text{lb-wt}}{4.448 N}$

$= 9422 \text{ lb-wt}$

$\boxed{F_1 = 9420 \text{ lb-wt}}$

$F_2 = \frac{P}{V_2}$

$= \frac{P}{5V_1}$

$= \frac{F_1}{5}$

$\boxed{F_2 = 1880 \text{ lb-wt}}$

$F_3 = \frac{P}{V_3}$

$= \frac{P}{3V_2}$

$= \frac{F_2}{3}$

$\boxed{F_3 = 630 \text{ lb-wt}}$

2-3

$$F(t) = \frac{P_0 \delta t}{\pi} \frac{1}{(t-t_0)^2 + (\delta t)^2}, \quad -\infty < t < \infty$$

$$I = \int_{t_0}^{t} F\,dt$$

$$= \int_{-\infty}^{\infty} \frac{P_0 \delta t}{\pi} \frac{dt}{(t-t_0)^2 + (\delta t)^2} \qquad \{\delta t \text{ is fixed}\}$$

$$= \frac{P_0}{\pi \delta t} \int_{-\infty}^{\infty} \frac{dt}{\left(\frac{t-t_0}{\delta t}\right)^2 + 1} \qquad u = \frac{t-t_0}{\delta t}$$

$$= \frac{P_0}{\pi} \int_{u_1}^{u_2} \frac{du}{u^2+1} \qquad du = \frac{dt}{\delta t}, \quad dt = du\,\delta t$$

$$= \frac{P_0}{\pi} \int_{\theta_1}^{\theta_2} \frac{\sec^2\theta\, d\theta}{\tan^2\theta + 1} \qquad u = \tan\theta, \quad du = \sec^2\theta\, d\theta$$

$$= \frac{P_0}{\pi} \int_{\theta_1}^{\theta_2} d\theta$$

$$= \frac{P_0}{\pi} (\theta_2 - \theta_1)$$

$$= \frac{P_0}{\pi} \left(\tan^{-1} u_2 - \tan^{-1} u_1\right)$$

$$= \frac{P_0}{\pi} \left(\tan^{-1}(\infty) - \tan^{-1}(-\infty)\right)$$

$$= \frac{P_0}{\pi} \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right)$$

$$\boxed{I = P_0}$$

For v_0 at $t = -\infty$

$$P_f - P_0 = \int_{-\infty}^{\infty} F\,dt = P_0$$

$$m v_f - m v_0 = P_0 \longrightarrow \boxed{v_{f\,(t=\infty)} = \frac{P_0}{m} + v_0}$$

2-5

$F(t) = F_0 \sin^2 \omega t$

$v_0 = v(0) = 0$

$x_0 = x(t) = 0$

$m\ddot{x} = F$

$\ddot{x} = \frac{F_0}{m} \sin^2 \omega t$

$v - v_0 = \int dv = \int \frac{F_0}{m} \sin^2 \omega t \, dt$

$= \int_0^t \frac{F_0}{m} \frac{1}{2}(1 - \cos 2\omega t') dt'$

$= \frac{F_0}{2m} \left[t - \int_0^t \cos 2\omega t' \, dt' \right]$

$= \frac{F_0}{2m} \left[t - \frac{1}{2\omega} \sin 2\omega t' \Big|_0^t \right]$

$= \frac{F_0}{2m} \left[t - \frac{1}{2\omega} \sin 2\omega t \right]$

$v_0 = v(0) = 0$

$\boxed{v = \frac{F_0}{2m} \left[t - \frac{1}{2\omega} \sin 2\omega t \right]}$

$\frac{dx}{dt} = v$

$x - x_0 = \int v \, dt$

$= \frac{F_0}{2m} \int_0^t \left[t' - \frac{1}{2\omega} \sin 2\omega t' \right] dt'$

$= \frac{F_0}{2m} \left[\frac{t'^2}{2} + \frac{1}{4\omega^2} \cos 2\omega t' \right] \Big|_0^t$

$x_0 = -\frac{F_0}{8m\omega^2}$

$x = \frac{F_0}{2m} \left[\frac{t^2}{2} + \frac{1}{4\omega^2} \cos 2\omega t \right] - \frac{F_0}{8m\omega^2}$

$\boxed{x = \frac{F_0}{8m\omega^2} \left[2\omega^2 t^2 - 1 + \cos 2\omega t \right]}$

2-7

$$F(t) = \begin{cases} 0, & t < t_0 \\ \frac{P_0}{\delta t}, & t_0 \leq t \leq t_0 + \delta t \\ 0, & t > t_0 + \delta t \end{cases}$$

Find $v(t)$ and $x(t)$

$$m \frac{dv}{dt} = F(t) \qquad v(0) = V_0$$

i) $t < t_0$

$$m \frac{dv}{dt} = 0 \quad , v(0) = V_0$$

$$dV = 0$$

$$\int_{V_0}^{V} dV = 0 \quad \text{or} \quad \int dV = 0 \rightarrow V(t) + C = 0 \rightarrow V(0) + C = 0 \rightarrow V_0 + C = 0 \rightarrow C = V_0 \Rightarrow$$

$$V - V_0 = 0$$

$$V - V_0 = 0$$

$$\boxed{V(t) = V_0}$$

$$\frac{dx}{dt} = V_0$$

$$\int_{x_0=0}^{x} dx = \int_0^t V_0 \, dt \quad x(0) = 0$$

$$\boxed{x = V_0 t}$$

ii) $t_0 \le t \le t_0 + \delta t$

$$m\frac{dv}{dt} = \frac{P_0}{\delta t}$$

$$dv = \frac{P_0}{m\delta t}$$

$$\int_{v_0}^{v} dv = \int_{t_0}^{t} \frac{P_0}{m\delta t}\, dt$$

$$v - v_0 = \frac{P_0}{m\delta t}(t - t_0)$$

$$\boxed{v = v_0 + \frac{P_0}{m\delta t}(t - t_0)}$$

$$\frac{dx}{dt} = v_0 + \frac{P_0}{m\delta t}(t - t_0)$$

$$dx = \left(v_0 + \frac{P_0}{m\delta t}(t - t_0)\right)dt$$

$$\int_{x_0}^{x} dx = \int_{t_0}^{t}\left(v_0 + \frac{P_0}{m\delta t}(t - t_0)\right)dt$$

$$x - x_0 = v_0 t - v_0 t_0 + \frac{P_0}{2m\delta t}(t - t_0)^2 \Big|_{t_0}^{t}$$

$x - x_0 = v_0 t - v_0 t_0 + \frac{P_0}{2m\delta t}(t - t_0)^2$, $x_0 = v_0 t_0$, the start point of the interval.

$$x - v_0 t_0 = v_0 t - v_0 t_0 + \frac{P_0}{2m\delta t}(t - t_0)^2$$

$$\boxed{x = v_0 t + \frac{P_0}{2m\delta t}(t - t_0)^2}$$

iii) $t > t_0 + \delta t$

$$m\frac{dv}{dt} = 0$$

$$dv = 0$$

$$\int_{v_0'}^{v} dv = 0$$

$$v - v_0' = 0 \qquad v_0' = v(t_0 + \delta t) \text{ at end of previous interval}$$

$$v = v_0' \qquad = v_0 + \frac{p_0}{m\delta t}(t - t_0)\Big|_{t_0 + \delta t}$$

$$\boxed{v = v_0 + \frac{p_0}{m}} \qquad = v_0 + \frac{p_0}{m\delta t}(\delta t)$$

$$v_0' = v_0 + \frac{p_0}{m}$$

$$\frac{dx}{dt} = v_0 + \frac{p_0}{m}$$

$$dx = \left(v_0 + \frac{p_0}{m}\right)dt$$

$$\int_{x_0'}^{x} dx = \int_{t_0+\delta t}^{t} \left(v_0 + \frac{p_0}{m}\right)dt$$

$$x - x_0' = v_0 t - v_0(t_0 + \delta t) + \frac{p_0}{m}t - \frac{p_0}{m}(t_0 + \delta t) \qquad x_0'\Big|_{t_0+\delta t} = v_0(t_0 + \delta t) + \frac{p_0}{2m\delta t}(t_0 + \delta t - t_0)^2$$

$$x - v_0(t_0 + \delta t) - \frac{p_0\delta t}{2m} = v_0 t - v_0(t_0 + \delta t) + \frac{p_0}{m}t - \frac{p_0}{m}(t_0 + \delta t) \qquad = v_0(t_0 + \delta t) + \frac{p_0}{2m}\delta t$$

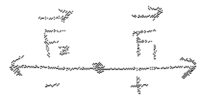
$$x = v_0 t + \frac{p_0\delta t}{2m} + \frac{p_0}{m}t - \frac{p_0}{m}(t_0 + \delta t)$$

$$x = v_0 t + \frac{p_0\delta t}{2m} + \frac{p_0}{m}t - \frac{p_0}{m}t_0 - \frac{p_0\delta t}{m}$$

$$\boxed{x = v_0 t - \frac{p_0\delta t}{2m} + \frac{p_0}{m}(t - t_0)}$$

2-9

$$\Sigma F = Ma \qquad\qquad M = \frac{(10)(160)}{32} = 50 \text{ slugs}$$



$$M\frac{d\vec{v}}{dt} = \vec{F_1} + \vec{F}$$

$$\vec{F_1} = 5F_1 \quad F_2 = -5F_2 \quad , F_1 = 200\,e^{-t/10} , \; F_2 = 200\,e^{-t/20}$$

$$M\frac{d\vec{v}}{dt} = 1000\,e^{-t/10} - 1000\,e^{-t/20}$$

$$dv = \frac{1000}{M}\left(e^{-t/10} - e^{-t/20}\right)dt$$

$$\int_0^v dv = \frac{1000}{M}\int_0^t \left(e^{-t/10} - e^{-t/20}\right)dt$$

$$V = \frac{1000}{M}\left[-10e^{-t/10} + 20e^{-t/20}\right]_0^t$$

$$= 20\left[-10e^{-t/10} + 20e^{-t/20} - (-10+20)\right]$$

$$= 20\left[20e^{-t/20} - 10e^{-t/10} - 10\right]$$

$$\frac{dx}{dt} = 20\left[20e^{-t/20} - 10e^{-t/10} - 10\right]$$

$$\int_0^x dx = 20\int_0^t \left[20e^{-t/20} - 10e^{-t/10} - 10\right]dt$$

$$X = 20\left[-400e^{-t/20} + 100e^{-t/10} - 10t\right]_0^t$$

$$= 200\left[-40e^{-t/20} + 10e^{-t/10} - t\right]_0^t$$

$$= 200\left[-40e^{-t/20} + 10e^{-2t/20} - t - (-40+10)\right]$$

$$= 200\left[10e^{-2t/20} - 40e^{-t/20} - t + 30\right]$$

$$10y^2 - 40y + 30 = 10(y^2 - 4y + 3) = 10(y-1)(y-3) = 10(1-y)(3-y)$$

$$y = e^{-t/20}$$

$$\boxed{X = -200t + 2000\left(1 - e^{-t/20}\right)\left(3 - e^{-t/20}\right)}$$

$$\boxed{V_{t=\infty} = \left.\frac{dx}{dt}\right|_{t=\infty} = -200 \text{ ft/s}}$$

$\boxed{V_{t=\infty} \text{ is unreasonable because F is independent of } v}$

2-11

$$m\frac{dv}{dt} = -be^{\alpha v}$$

$$\frac{dv}{e^{\alpha v}} = -\frac{b}{m}dt$$

$$e^{-\alpha v}dv = -\frac{b}{m}dt$$

$$\int_{v_0}^{v} e^{-\alpha v}dv = -\frac{b}{m}t$$

$$-\frac{1}{\alpha}e^{-\alpha v}\Big|_{v_0}^{v} = -\frac{b}{m}t$$

$$-\frac{1}{\alpha}\left[e^{-\alpha v} - e^{-\alpha v_0}\right] = -\frac{b}{m}t$$

$$e^{-\alpha v} - e^{-\alpha v_0} = \frac{\alpha b}{m}t$$

Time to stop, $v=0$

$$1 - e^{-\alpha v_0} = \frac{\alpha b}{m}t$$

$$\boxed{t_s = \frac{m(1-e^{-\alpha v_0})}{\alpha b}}$$

$$e^{-\alpha v} = \frac{\alpha b}{m}t + e^{-\alpha v_0}$$

$$-\alpha v = \ln\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)$$

$$\frac{dx}{dt} = -\frac{1}{\alpha}\ln\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)$$

$$\int_0^x dx = -\frac{1}{\alpha}\int_0^t \ln\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)dt$$

$$u = \frac{\alpha b}{m}t + e^{-\alpha v_0}$$

$$du = \frac{\alpha b}{m}dt$$

$$\frac{m}{\alpha b}du = dt$$

$$x = -\frac{m}{\alpha^2 b}\left[\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)\left(\ln\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right) - 1\right)\right]_0^t$$

$$\int \ln\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)dt = \frac{m}{\alpha b}\int \ln u\, du$$

$$= \frac{m}{\alpha b}(u\ln u - u)$$

$$X = \frac{-m}{\alpha^2 b}\left[\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)\left(\ln\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right) - 1\right) - \left(e^{-\alpha v_0}(-\alpha v_0 - 1)\right)\right]$$

$$= \frac{-m}{\alpha^2 b}\left[\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)\left(\ln\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)\right) - \frac{\alpha b}{m}t - e^{-\alpha v_0} + \alpha v_0 e^{-\alpha v_0} + e^{-\alpha v_0}\right]$$

$$\boxed{X = \frac{-m}{\alpha^2 b}\left[\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)\left(\ln\left(\frac{\alpha b}{m}t + e^{-\alpha v_0}\right)\right) - \frac{\alpha b}{m}t + \alpha v_0 e^{-\alpha v_0}\right]}$$

$$X_s = \frac{m}{\alpha^2 b}\left[\frac{\alpha b}{m}t_s - \alpha v_0 e^{-\alpha v_0} - \left(\frac{\alpha b}{m}t_s + e^{-\alpha v_0}\right)\ln\left(\frac{\alpha b}{m}t_s + e^{-\alpha v_0}\right)\right]$$

$$\frac{\alpha b}{m}t_s = \frac{\alpha b}{m}\left[\frac{m(1 - e^{-\alpha v_0})}{\alpha b}\right]$$

$$= (1 - e^{-\alpha v_0})$$

$$X_s = \frac{m}{\alpha^2 b}\left[1 - e^{-\alpha v_0} - \alpha v_0 e^{-\alpha v_0} - \left(1 - e^{-\alpha v_0} + e^{-\alpha v_0}\right)\ln\left(1 - e^{-\alpha v_0} + e^{-\alpha v_0}\right)\right]$$

$$\boxed{X_s = \frac{m}{\alpha^2 b}\left[1 - e^{-\alpha v_0} - \alpha v_0 e^{-\alpha v_0}\right]}$$

2-13

$$m\frac{dv}{dt} = F_0 - bv^2 \qquad x_0 = v_0 = 0 \text{ at } t_0 = 0$$

$$\frac{dv}{F_0 - bv^2} = \frac{dt}{m}$$

$$\int_0^v \frac{dv}{F_0 - bv^2} = \int_0^t \frac{dt}{m}$$

$$\int_0^v \frac{dv}{F_0 - bv^2} = \frac{t}{m} \qquad v^2 = \frac{F_0}{b}\tanh^2 y$$

$$v = \sqrt{\frac{F_0}{b}}\tanh y \rightarrow y = \tanh^{-1}\left(v\sqrt{\frac{b}{F_0}}\right)$$

$$\int_0^y \frac{\sqrt{\frac{F_0}{b}}\,\mathrm{sech}^2 y\, dy}{F_0(1-\tanh^2 y)} = \frac{t}{m} \qquad dv = \sqrt{\frac{F_0}{b}}\,\mathrm{sech}^2 y\, dy$$

$$\frac{1}{\sqrt{F_0 b}}\int_0^y dy = \frac{t}{m} \qquad \{1 - \tanh^2 y = \mathrm{sech}^2 y\}$$

$$\frac{1}{\sqrt{F_0 b}}\left[\tanh^{-1}\left(v\sqrt{\frac{b}{F_0}}\right)\right] = \frac{t}{m}$$

$$\tanh^{-1}\left(v\sqrt{\frac{b}{F_0}}\right) = \frac{t}{m}\sqrt{F_0 b}$$

$$v\sqrt{\frac{b}{F_0}} = \tanh\left(\frac{t}{m}\sqrt{F_0 b}\right)$$

$$\boxed{v = \sqrt{\frac{F_0}{b}}\tanh\left(\frac{t}{m}\sqrt{F_0 b}\right)}$$

2-15 $\quad v_0 = 0,\ t_0 = 0,\ F = \frac{P}{v} - bv$

$$m\frac{dv}{dt} = \frac{P}{v} - bv$$

$$= \frac{-bv^2 + P}{v}$$

$$\frac{v\,dv}{-bv^2 + P} = \frac{dt}{m}$$

$$\frac{v\,dv}{-v^2 + \frac{P}{b}} = \frac{b\,dt}{m}$$

$$\int_0^v \frac{v\,dv}{-v^2 + \frac{P}{b}} = \int_0^t \frac{b\,dt}{m}$$

$$-\frac{1}{2}\ln\left|-\frac{b}{P}v^2 + 1\right| = \frac{bt}{m}$$

$$\ln\left|-\frac{b}{P}v^2 + 1\right| = \frac{-2bt}{m}$$

$$-\frac{b}{P}v^2 + 1 = e^{-\frac{2bt}{m}}$$

$$-\frac{b}{P}v^2 = -1 + e^{-\frac{2bt}{m}}$$

$$\frac{b}{P}v^2 = 1 - e^{-\frac{2bt}{m}}$$

$$v^2 = \frac{P}{b}\left(1 - e^{-\frac{2bt}{m}}\right)$$

$$\boxed{v = \sqrt{\frac{P}{b}}\left(1 - e^{-\frac{2bt}{m}}\right)^{1/2}}$$

let $u = -v^2 + \frac{P}{b}$

$du = -2v\,dv$

$$-\frac{du}{2} = v\,dv \Rightarrow \int_0^v \frac{v\,dv}{-v^2 + \frac{P}{b}} = -\frac{1}{2}\int_{\frac{P}{b}}^{-v^2 + \frac{P}{b}} \frac{du}{u}$$

$$= -\frac{1}{2}\ln\left|\frac{-v^2 + \frac{P}{b}}{\frac{P}{b}}\right|$$

$$= -\frac{1}{2}\ln\left|-\frac{b}{P}v^2 + 1\right|$$

2-17

$$F = (\mp) b v^n \quad (n \neq 1)$$

$$x_0 = 0$$

$$v(0) = v_0$$

$$m\ddot{x} = F$$

$$m\frac{dv}{dt} = F$$

$$m\, dv = -b v^n dt$$

$$\frac{dv}{v^n} = -\frac{b}{m} dt$$

$$\int \frac{dv}{v^n} = -\frac{b}{m} t \quad \{t_0 = 0\}$$

$$\int_{v_0}^{v} v'^{-n} dv' = -\frac{b}{m} t$$

$$\left.\frac{v^{-n+1}}{-n+1}\right|_{v_0}^{v} = -\frac{b}{m} t$$

$$\frac{v^{1-n}}{1-n} - \frac{v_0^{1-n}}{1-n} = -\frac{b}{m} t$$

$$v^{1-n} - v_0^{1-n} = -\frac{b}{m} t(1-n)$$

$$v^{1-n} = -\frac{b}{m} t(1-n) + v_0^{1-n}$$

$$(1-n)\ln v = \ln\left[-\frac{b}{m} t(1-n) + v_0^{1-n}\right]$$

$$\ln v = \frac{\ln\left[-\frac{b}{m} t(1-n) + v_0^{1-n}\right]}{1-n}$$

$$\ln v = \ln\left[-\frac{b}{m} t(1-n) + v_0^{1-n}\right]^{\frac{1}{1-n}}$$

$$\boxed{v = \left[v_0^{1-n} - (1-n)\frac{b}{m} t\right]^{\frac{1}{1-n}}}$$

$$dx = \left[v_0^{1-n} - (1-n)\frac{b}{m} t\right]^{\frac{1}{1-n}} dt$$

$$x = \int_0^t \left[v_0^{1-n} - (1-n)\frac{b}{m} t'\right]^{\frac{1}{1-n}} dt'$$

$$u = v_0^{1-n} - (1-n)\frac{b}{m} t'$$

$$du = -(1-n)\frac{b}{m} dt'$$

$$dt' = \frac{du}{-(1-n)\frac{b}{m}}$$

$$= \int_{u_0}^{u'} -\frac{u^{\frac{1}{1-n}}}{(1-n)\frac{b}{m}} du$$

$$= \frac{-m}{(1-n)b} \left.\frac{u^{1+\frac{1}{1-n}}}{1+\frac{1}{1-n}}\right|_{u_0}^{u'}$$

$$x = \frac{-m}{(1-n)b} \left. \frac{u^{\frac{2-n}{1-n}}}{\frac{2-n}{1-n}} \right|_{u_0}^{u'}$$

$$= \frac{-m}{(1-n)b} (1-n) \left. \frac{u^{\frac{2-n}{1-n}}}{2-n} \right|_{u_0}^{u'}$$

$$= \frac{-m}{(2-n)b} \left. u^{\frac{2-n}{1-n}} \right|_{u_0}^{u'}$$

$$= \frac{-m}{(2-n)b} \left. \left[v_0^{1-n} - (1-n)\frac{b}{m}t' \right]^{\frac{2-n}{1-n}} \right|_0^t$$

$$= \frac{-m}{(2-n)b} \left(\left[v_0^{1-n} - (1-n)\frac{b}{m}t \right]^{\frac{2-n}{1-n}} - \left[v_0^{1-n} \right]^{\frac{2-n}{1-n}} \right)$$

$$= \frac{-m}{(2-n)b} \left[-v_0^{2-n} + \left[v_0^{1-n} - (1-n)\frac{b}{m}t \right]^{\frac{2-n}{1-n}} \right]$$

$$\boxed{x = \frac{m}{(2-n)b} \left[v_0^{2-n} - \left[v_0^{1-n} - (1-n)\frac{b}{m}t \right]^{\frac{2-n}{1-n}} \right]}$$

Time to stop:

$$v = 0$$

$$0 = \left[v_0^{1-n} - (1-n)\frac{b}{m}t_s \right]^{\frac{1}{1-n}}$$

$$(1-n)\frac{b}{m}t_s = v_0^{1-n}$$

$$\boxed{t_s = v_0^{1-n}\frac{m}{b(1-n)} \quad , n > 1}$$

Distance required to stop:

$$x_s = \frac{m}{(2-n)b} \left[v_0^{2-n} - \left[v_0^{1-n} - (1-n)\frac{b}{m}t_s \right]^{\frac{2-n}{1-n}} \right]$$

$$\boxed{x_s = \frac{m}{(2-n)b} \left[v_0^{2-n} \right]}$$

2-19

$$m\frac{dv}{dt} = \frac{k}{x^3}, \quad t_0 = 0, \; v_0 = 0$$

$$m\frac{dv}{dt} = F$$

$$\sqrt{\frac{m}{2}}\int_{x_0}^{x}\left[E - V(x)\right]^{-1/2}dx = t - t_0 \quad (2.46)$$

$$\sqrt{\frac{m}{2}}\int_{x_0}^{x}\left[\frac{k}{2}x_0^{-2} - \frac{k}{2}x^{-2}\right]^{-1/2}dx = t$$

$$\sqrt{\frac{m}{2}}\sqrt{\frac{2}{k}}\int_{x_0}^{x}\left[x_0^{-2} - x^{-2}\right]^{-1/2}dx = t$$

$$\sqrt{\frac{m}{k}}\,x_0\left[x^2 - x_0^2\right]^{1/2}\Big|_{x_0}^{x} = t$$

$$\sqrt{\frac{m}{k}}\,x_0\left(x^2 - x_0^2\right)^{1/2} = t$$

$$\frac{m}{k}x_0^2\left(x^2 - x_0^2\right) = t^2$$

$$x^2 - x_0^2 = \frac{kt^2}{mx_0^2}$$

$$x^2 = \frac{kt^2}{mx_0^2} + x_0^2$$

$$\boxed{x = \left[x_0^2 + \frac{kt^2}{mx_0^2}\right]^{1/2}}$$

$$V(x) = -\int_{\infty}^{x}F\,dx$$

$$= -k\int_{\infty}^{x}x^{-3}dx$$

$$V(x) = \frac{k}{2}x^{-2}$$

$$E = T_0 + V(x_0)$$

$$= 0 + \frac{k}{2}x_0^{-2}$$

$$E = \frac{k}{2}x_0^{-2}$$

$$\int\left[x_0^{-2} - x^{-2}\right]^{-1/2}dx = \int\left[\frac{1}{x_0^2} - \frac{1}{x^2}\right]^{-1/2}dx$$

$$= \int\left[\frac{x^2 - x_0^2}{x_0^2x^2}\right]^{-1/2}dx$$

$$= x_0\int\frac{x\,dx}{(x^2 - x_0^2)^{1/2}}$$

let $u = x^2 - x_0^2$

$du = 2x\,dx$

$\frac{du}{2} = x\,dx$

$$= \frac{x_0}{2}\int u^{-1/2}du$$

$$= \frac{x_0}{2}\frac{2}{1}u^{1/2} + C$$

$$= x_0\left(x^2 - x_0^2\right)^{1/2} + C$$

2-21

$$V = ax^2 - bx^3$$

$$\frac{1}{2}mv^2 + V(x) = \frac{1}{2}mv_0^2 + V(x_0)$$
$$= E$$
$$v^2 = \frac{2}{m}(E - V(x))$$

$$E = \frac{1}{2}mv_0^2 + V(0)$$
$$E = \frac{1}{2}mv_0^2$$

a) Find F

$$F = -\frac{dV}{dx}$$
$$= -\frac{d}{dx}(ax^2 - bx^3)$$
$$\boxed{F = -2ax + 3bx^2}$$

b) Find V_c, $x_0 = 0$

$F = 0$ {where $V(x)$ is a maximum}

$$-2ax + 3bx^2 = 0$$
$$x(3bx - 2a) = 0$$
$$x = 0, \quad 3bx - 2a = 0$$
$$x_c = \frac{2a}{3b}$$

$$v^2 = \frac{2}{m}(E - V(x))$$
$$= \frac{2}{m}\left(\frac{1}{2}mv_0^2 - ax^2 + bx^3\right)$$
$$0 = \frac{2}{m}\left(\frac{1}{2}mV_c^2 - ax^2 + bx^3\right)$$
$$= V_c^2 - \frac{2}{m}ax_c^2 + \frac{2}{m}bx_c^3$$
$$V_c^2 = \frac{2}{m}(ax_c^2 - bx_c^3)$$
$$= \frac{2}{m}\left(a\left(\frac{2a}{3b}\right)^2 - b\left(\frac{2a}{3b}\right)^3\right)$$
$$= \frac{2}{m}\left[\frac{4a^2}{9b^2} - \frac{8a^3}{27b^2}\right]$$
$$= \frac{2}{m}\left[\frac{12a^3}{27b^2} - \frac{8a^3}{27b^2}\right]$$
$$\boxed{V_c^2 = \frac{8a^3}{27b^2m}}$$

2-23

$$F = -kx + ax^{-3}$$

a) Find $V(x)$ and $x(t)$

$$V(x) = -\int F\,dx$$

$$= -\int -kx\,dx - \int ax^{-3}\,dx$$

$$= \frac{k}{2}x^2 + \frac{a}{2}x^{-2}$$

$$V(x) = \frac{1}{2}(kx^2 + ax^{-2})$$

$$\sqrt{\frac{m}{2}}\int_{x_0}^{x}\left[E - V(x)\right]^{-1/2}dx = t \quad , \quad t_0 = 0$$

$$\sqrt{\frac{m}{2}}\int_{x_0}^{x}\left[E - \frac{1}{2}(kx^2 + ax^{-2})\right]^{-\frac{1}{2}}dx = t$$

$$\sqrt{\frac{m}{2}}\sqrt{\frac{2}{1}}\int_{x_0}^{x}\left[2E - (kx^2 + ax^{-2})\right]^{-\frac{1}{2}}dx =$$

$$\sqrt{m}\int_{x_0}^{x}\left[2E - kx^2 - \frac{a}{x^2}\right]^{-\frac{1}{2}}dx =$$

$$\sqrt{m}\int_{x_0}^{x}\left[2E - \frac{kx^4 - a}{x^2}\right]^{-\frac{1}{2}}dx =$$

$$\sqrt{m}\int_{x_0}^{x}\left[\frac{2Ex^2 - kx^4 - a}{x^2}\right]^{-\frac{1}{2}}dx =$$

$$\sqrt{m}\int_{x_0}^{x}\frac{x\,dx}{(2Ex^2 - kx^4 - a)^{1/2}} = t$$

$$-kx^4 + 2Ex^2 - a = -k\left(x^4 - \frac{2E}{k}x^2 + \frac{a}{k}\right)$$

$$= -k\left(x^4 - \frac{2E}{k}x^2 + \frac{E^2}{k^2} + \frac{a}{k} - \frac{E^2}{k^2}\right)$$

$$= -k\left[\left(x^2 - \frac{E}{k}\right)^2 - \left(\frac{E^2}{k^2} - \frac{a}{k}\right)\right]$$

$$= k\left[\frac{E^2 - ak}{k^2} - \left(x^2 - \frac{E}{k}\right)^2\right]$$

$$\sqrt{m}\int_{x_0}^{x}\frac{x\,dx}{\left[K\left(\frac{E^2-ak}{k^2}-\left(x^2-\frac{E}{k}\right)^2\right)\right]^{1/2}}=t$$

$$\sqrt{\frac{m}{k}}\int_{x_0}^{x}\frac{x\,dx}{\left[\frac{E^2-ak}{k^2}-\left(x^2-\frac{E}{k}\right)^2\right]^{1/2}}=t$$

$$\frac{1}{2}\sqrt{\frac{m}{k}}(\theta-\theta_0)=t$$

$$\theta=2t\sqrt{\frac{k}{m}}+\theta_0$$

$$x^2=\frac{E}{k}+\frac{\sqrt{E^2-ak}}{k}\sin\theta$$

$$\boxed{x^2=\frac{E}{k}+\frac{\sqrt{E^2-ak}}{k}\sin\left[2t\sqrt{\frac{k}{m}}+\theta_0\right]}$$

$$\sqrt{\frac{m}{k}}\int_{x_0}^{x}\frac{x\,dx}{\left[\frac{E^2-ak}{k^2}-\left(x^2-\frac{E}{k}\right)^2\right]^{1/2}}$$

let $x^2-\frac{E}{k}=\frac{\sqrt{E^2-ak}}{k}\sin\theta$

$$2x\,dx=\frac{\sqrt{E^2-ak}}{k}\cos\theta\,d\theta$$

$$\frac{1}{2}\sqrt{\frac{m}{k}}\int_{\theta_0}^{\theta}\frac{\frac{\sqrt{E^2-ak}}{k}\cos\theta\,d\theta}{\left[\frac{E^2-ak}{k^2}-\frac{E^2-ak}{k^2}\sin^2\theta\right]^{1/2}}=$$

$$\frac{1}{2}\sqrt{\frac{m}{k}}\int_{\theta_0}^{\theta}\frac{\cos\theta\,d\theta}{(1-\sin^2\theta)^{1/2}}=$$

$$\frac{1}{2}\sqrt{\frac{m}{k}}\int_{\theta_0}^{\theta}d\theta=\frac{1}{2}\sqrt{\frac{m}{k}}(\theta-\theta_0)$$

2-25

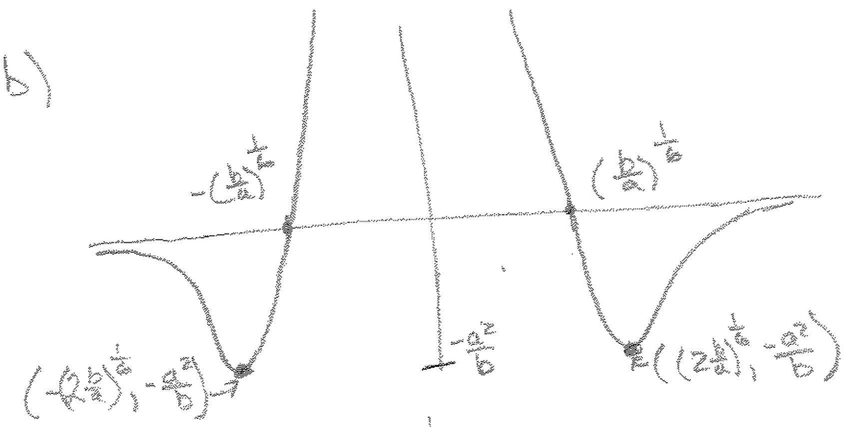
$$V(x) = -\frac{a}{x^6} + \frac{b}{x^{12}}$$

a) $F = -\frac{dV}{dx}$

$$= -\frac{d}{dx}\left(-ax^{-6} + bx^{-12}\right)$$

$$\boxed{F = -6ax^{-7} + 12bx^{-13}}$$

b)



$\boxed{V(x_0) < 0 \rightarrow \text{oscillates}}$

Zeros: $0 = -\frac{a}{x^6} + \frac{b}{x^{12}}$

$$\frac{a}{x^6} = \frac{b}{x^{12}}$$

$$\frac{x^6}{a} = \frac{x^{12}}{b}$$

$$\frac{x^{12}}{b} - \frac{x^6}{a} = 0$$

$$x^6\left(\frac{x^6}{b} - \frac{1}{a}\right) = 0$$

$$\frac{x^6}{b} - \frac{1}{a} = 0$$

$$\frac{x^6}{b} = \frac{1}{a}$$

$$x^6 = \frac{b}{a}$$

$$x = \pm\sqrt[6]{\frac{b}{a}}$$

max, min

$$0 = -6ax^{-7} + 12bx^{-13}$$

$$6ax^{-7} = 12bx^{-13}$$

$$\frac{6a}{x^7} = \frac{12b}{x^{13}}$$

$$\frac{x^7}{6a} = \frac{x^{13}}{12b}$$

$$\frac{x^{13}}{12b} - \frac{x^7}{6a} = 0$$

$$x^7\left(\frac{x^6}{12b} - \frac{1}{6a}\right) = 0$$

$$\frac{x^6}{12b} = \frac{1}{6a}$$

$$x^6 = 2\frac{b}{a}$$

$$x = \pm\sqrt[6]{2\frac{b}{a}}$$

$$V\left(\pm\sqrt[6]{\frac{2b}{a}}\right) = -\frac{a}{2\frac{b}{a}} + \frac{b}{4\frac{b^2}{a^2}}$$

$$= -\frac{a^2}{2b} + \frac{a^2}{4b}$$

$$= \frac{-2a^2 + a^2}{4b}$$

$$= \frac{-a^2}{b}$$

c) From b) equilibrium distance is

$$x_{eq} = \pm\left(\frac{2b}{a}\right)^{1/6}$$

To determine period of small oscillations about x_{eq}

$$V(x) = V(x_{eq}) + \left(\frac{dV}{dx}\right)_{x_{eq}}(x-x_{eq}) + \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_{x_{eq}}(x-x_{eq})^2 + \frac{1}{6}\left(\frac{d^3V}{dx^3}\right)_{x_{eq}}(x-x_{eq})^3 + \cdots$$

{Taylor expansion about x_{eq}}

$\left(\frac{dV}{dx}\right)_{x_{eq}} = 0$, can drop $V(x_{eq})$ without effecting physical results.

$$V(x) = \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_{x_{eq}}(x-x_{eq})^2$$

Letting $\left(\frac{d^2V}{dx^2}\right)_{x_{eq}} = k$ and $x' = x - x_{eq}$

$V(x') = \frac{1}{2}kx'^2 + \cdots$, since $x - x_{eq} \ll 1$, higher terms can be ignored.

So, $\omega = \sqrt{\frac{k}{m}}$.

$$k = \frac{d^2}{dx^2}\left(-\frac{a}{x^6} + \frac{b}{x^{12}}\right)_{x_{eq}}$$

$$= \frac{d}{dx}\left(6ax^{-7} - 12bx^{-13}\right)_{x_{eq}}$$

$$= \left(-42ax^{-8} + 156bx^{-14}\right)_{x_{eq} = \left(\frac{2b}{a}\right)^{1/6}}$$

$$= -42a\left(\frac{2b}{a}\right)^{-\frac{4}{3}} + 156b\left(\frac{2b}{a}\right)^{-\frac{7}{3}}$$

$$= -42a\left(\frac{a}{2b}\right)^{\frac{4}{3}} + 156b\left(\frac{a}{2b}\right)^{\frac{7}{3}}$$

$$= -42\frac{a^2}{2b}\left(\frac{a}{2b}\right)^{\frac{1}{3}} + 156\frac{a^2}{4b}\left(\frac{a}{2b}\right)^{\frac{1}{3}}$$

$$= -21\frac{a^2}{b}\left(\frac{a}{2b}\right)^{\frac{1}{3}} + 39\frac{a^2}{b}\left(\frac{a}{2b}\right)^{1/3}$$

$$= 18\frac{a^2}{b}\left(\frac{a}{2b}\right)^{\frac{1}{3}}$$

$$\omega = \sqrt{\frac{k}{m}}$$

$$2\pi f = \sqrt{\frac{k}{m}}$$

$$\frac{1}{2\pi f} = \sqrt{\frac{m}{k}}$$

$$T = \frac{1}{f} = 2\pi\sqrt{\frac{m}{k}}$$

$$T = 2\pi\left(\frac{m}{18\frac{a^2}{b}\left(\frac{a}{2b}\right)^{1/3}}\right)^{1/2}$$

$$= 2\pi \frac{m^{1/2}}{18^{1/2}\left(\frac{a^2}{b}\right)^{1/2}\left(\frac{a}{2b}\right)^{1/6}}$$

$$= \frac{2\pi}{3}\left[\frac{m^{1/2}}{2^{1/2}\left(\frac{a^2}{b}\right)^{1/2}\left(\frac{a}{2b}\right)^{1/6}}\right]$$

$$= \frac{2\pi}{3}\left[\frac{m^{3/6}}{2^{3/6}\left(\frac{a^2}{b}\right)^{3/6}\left(\frac{a}{2b}\right)^{1/6}}\right]$$

$$\boxed{T = \frac{2\pi}{3}\left[\frac{m^3 b^4}{4a^7}\right]^{1/6}}$$

2-27 $V(x) = \dfrac{-V_0 a^2 (a^2+x^2)}{8a^4+x^4}$

a) $F = -\dfrac{dV}{dx}$

$= -\dfrac{d}{dx}\left(\dfrac{-V_0 a^4 - V_0 a^2 x^2}{8a^4+x^4}\right)$

$= \dfrac{d}{dx}\left(\dfrac{V_0 a^4 + V_0 a^2 x^2}{8a^4+x^4}\right)$

$F = \dfrac{(8a^4+x^4)(2V_0 a^2 x) - (V_0 a^4 + V_0 a^2 x^2)(4x^3)}{(8a^4+x^4)^2}$

max and min of $V(x) \rightarrow \frac{dV}{dx} = 0$

$0 = \dfrac{d}{dx}\left(\dfrac{-V_0 a^2 (a^2+x^2)}{8a^4+x^4}\right)$

$= -V_0 a^2 \dfrac{d}{dx}\left(\dfrac{a^2+x^2}{8a^4+x^4}\right)$

$= \dfrac{(8a^4+x^4)2x - (a^2+x^2)4x^3}{(8a^4+x^4)^2}$

$= 16a^4x + 2x^5 - 4a^2x^3 - 4x^5$

$= -2x^5 - 4a^2x^3 + 16a^4x$

$= x^5 + 2a^2x^3 - 8a^4x$

$= x(x^4 + 2a^2x^2 - 8a^4)$

$\boxed{x=0}$, $x^4 + 2a^2x^2 - 8a^4 = 0$ $\quad y = x^2$

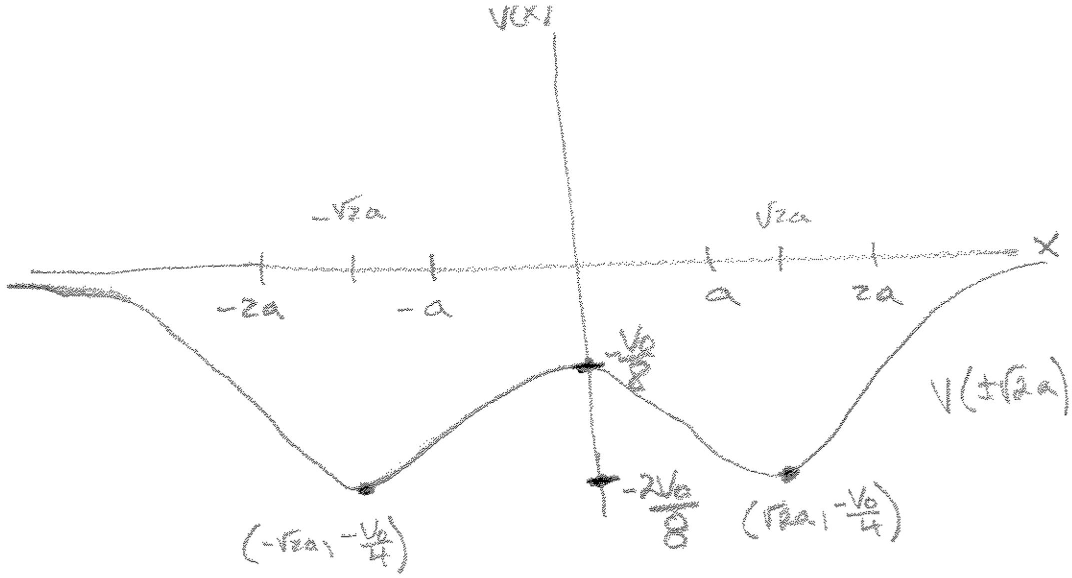
$y^2 + 2a^2y - 8a^4 = 0$ $\quad x^2 = -4a^2$ or $2a^2$

$y^2 + 2a^2y + a^4 = 8a^4 + a^4$ $\quad \boxed{x = \pm\sqrt{2}a}$

$(y+a^2)^2 = 9a^4$

$y + a^2 = \pm 3a^2$

$y = -4a^2$ or $2a^2$



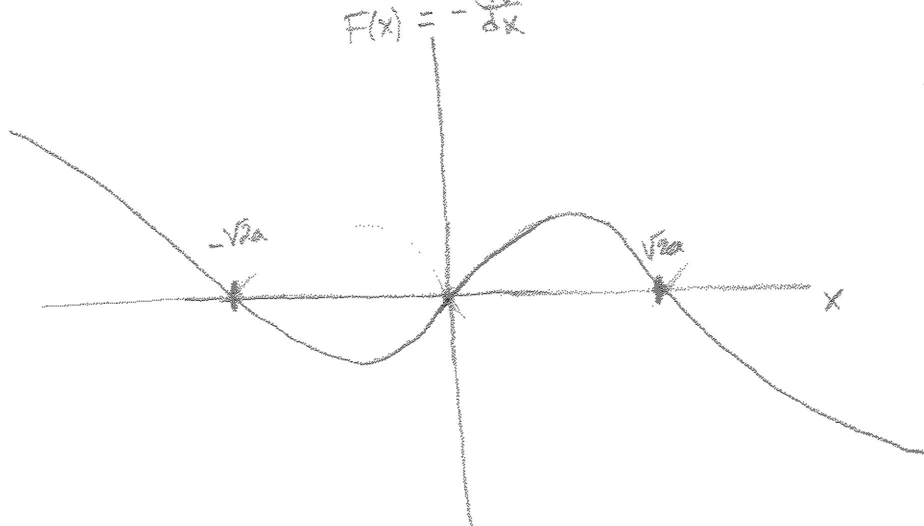
$$V(\pm\sqrt{2}a) = \frac{-V_0 a^2 (a^2 + 2a^2)}{8a^4 + 4a^4}$$

$$= \frac{-V_0 a^2 (3a^2)}{12a^4}$$

$$= -\frac{3V_0 a^4}{12a^4}$$

$$= -\frac{3V_0}{12}$$

$$= -\frac{V_0}{4}$$



b) $-\frac{V_0}{4} < V(x) < -\frac{V_0}{8}$ is oscillatory about $x_{eq} = \pm\sqrt{2}a$

$-\frac{V_0}{8} < x < 0$ is oscillatory about $x = 0$

Small oscillations are about $x_{eq} = \pm\sqrt{2}a$

$$V(x) = V(x_{eq}) + \left(\frac{dV}{dx}\right)_{x_{eq}}(x - x_{eq}) + \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_{x_{eq}}(x - x_{eq})^2 + \frac{1}{6}\left(\frac{d^3V}{dx^3}\right)_{x_{eq}}(x - x_{eq})^3 + \cdots$$

$$= \frac{1}{2}\left(\frac{d^2V}{dx^2}\right)_{x_{eq}}(x - x_{eq})^2 + \cdots$$

$$K = \left(\frac{d^2V}{dx^2}\right)_{x_{eq}}$$

$$\frac{dV}{dx} = -V_0 a^2 \frac{d}{dx}\left(\frac{a^2 + x^2}{8a^4 + x^4}\right)$$

$$\frac{dV}{dx} = -V_0 a^2 \left[\frac{(8a^4+x^4)2x - (a^2+x^2)4x^3}{(8a^4+x^4)^2} \right]$$

$$= -V_0 a^2 \left[\frac{16a^4x + 2x^5 - 4a^2x^3 - 4x^5}{(8a^4+x^4)^2} \right]$$

$$= -V_0 a^2 \left[\frac{-2x^5 - 4a^2x^3 + 16a^4x}{(8a^4+x^4)^2} \right]$$

$$\frac{d^2V}{dx^2} = -V_0 a^2 \left[\frac{(8a^4+x^4)^2(-10x^4 - 12a^2x^2 + 16a^4) - (-2x^5 - 4a^2x^3 + 16a^4x)2(8a^4+x^4)4x^3}{(8a^4+x^4)^4} \right]$$

$$= -V_0 a^2 (8a^4+x^4) \left[\frac{(8a^4+x^4)(-10x^4 - 12a^2x^2 + 16a^4) - 8x^3(-2x^5 - 4a^2x^3 + 16a^4x)}{(8a^4+x^4)^4} \right]$$

$$= -V_0 a^2 \left[\frac{(8a^4+x^4)(-10x^4 - 12a^2x^2 + 16a^4) + 16x^4(x^4 + 2a^2x^2 - 8a^4)}{(8a^4+x^4)^3} \right]$$

$$K = \left(\frac{d^2V}{dx^2} \right)_{x_{eq}} = -V_0 a^2 \left[\frac{(8a^4+4a^4)(-10(4a^4) - 12a^2(2a^2) + 16a^4) + 16x^4(4a^4 + 2a^2(2a^2) - 8a^4)}{(8a^4+4a^4)^3} \right]$$

$$x_{eq} = \pm\sqrt{2}a$$

$$= -V_0 a^2 \left[\frac{-40a^4 - 24a^4 + 16a^4}{(12a^4)^2} \right]$$

$$= -V_0 a^2 \left[\frac{-48a^4}{144a^8} \right]$$

$$= V_0 a^2 \left[\frac{a^4}{3a^8} \right]$$

$$K = \frac{V_0}{3a^2}$$

$$\omega = \sqrt{\frac{K}{m}}$$

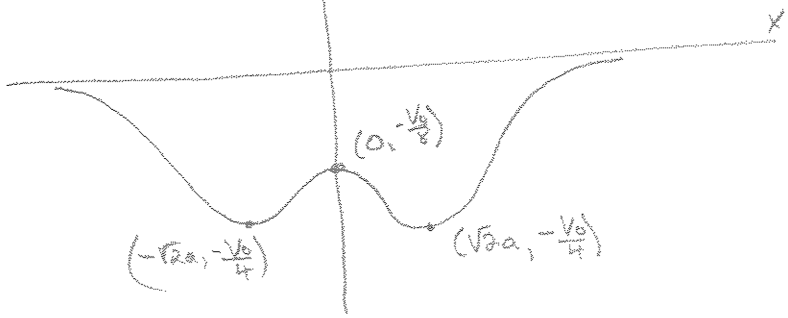
$$2\pi f = \sqrt{\frac{K}{m}}$$

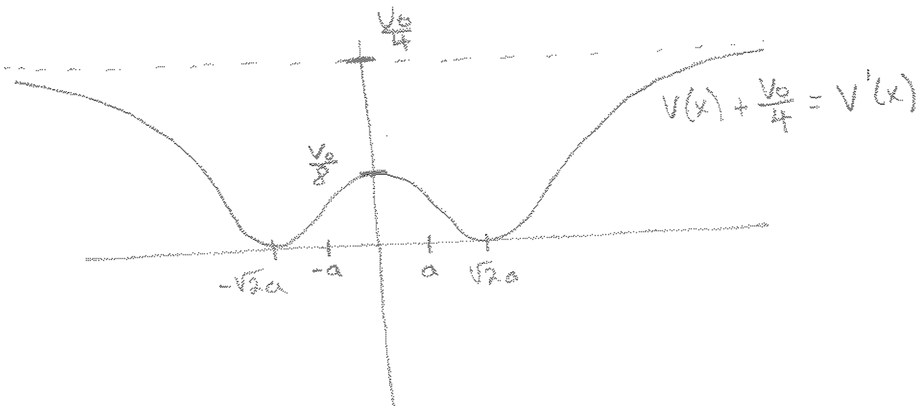
$$f = \frac{\left(\frac{V_0}{3a^2m} \right)^{1/2}}{2\pi}$$

$$\boxed{f = \left(\frac{V_0}{12\pi^2 a^2 m} \right)^{1/2}}$$

c)

$$V(x) = \frac{-V_0 a \ldots}{8a^4 + x^4}$$





$$E = \tfrac{1}{2} m v_0^2 + V(x \gg 0) + \frac{V_0}{4}$$

$$= \tfrac{1}{2} m v_0^2 + 0 + \frac{V_0}{4}$$

$$E = \tfrac{1}{2} m v_0^2 + \frac{V_0}{4}$$

$$V'(a) = V(a) + \frac{V_0}{4}$$

$$= \frac{-V_0 a^2(a^2 + a^2)}{8a^4 + a^4} + \frac{V_0}{4}$$

$$= \frac{-2V_0 a^4}{9a^4} + \frac{V_0}{4}$$

$$= \frac{-2V_0}{9} + \frac{V_0}{4} = \frac{-8V_0}{36} + \frac{9V_0}{36} = \frac{V_0}{36}$$

$$E = T_a + V'(a)$$

$$\tfrac{1}{2} m v_0^2 + \frac{V_0}{4} = T_a + V'(a)$$

$$\tfrac{1}{2} m v_0^2 + \frac{V_0}{4} - V'(a) = T_a$$

$$\tfrac{1}{2} m v_0^2 + \frac{V_0}{4} - \left(\frac{V_0}{36}\right) = T_a$$

$$\tfrac{1}{2} m v_0^2 + \frac{9V_0}{36} - \frac{V_0}{36} = T_a$$

$$\tfrac{1}{2} m v_0^2 + \frac{8V_0}{36} = T_a$$

$$\tfrac{1}{2} m v_0^2 + \frac{2V_0}{9} = T_a$$

$$\alpha > \frac{T_0}{T_a} \quad \{\alpha T_a > T_0,\ \alpha T_a = \text{kinetic energy removed by collision}\}$$

$$\alpha > \frac{\tfrac{1}{2} m v_0^2}{\tfrac{1}{2} m v_0^2 + \frac{2V_0}{9}}$$

$$\frac{1}{\alpha} < \frac{\tfrac{1}{2} m v_0^2 + \frac{2V_0}{9}}{\tfrac{1}{2} m v_0^2}$$

$$< 1 + \frac{4V_0}{9 m v_0^2}$$

$$\boxed{\alpha > \left[1 + \frac{4V_0}{9 m v_0^2}\right]^{-1}}$$ remains in well

Trapped in one side of well

$$T_a = \frac{1}{2} m v_0^2 + \frac{2V_0}{9}$$

Energy must be reduced by

$$\frac{V_0}{4} - \frac{V_0}{8} = \frac{V_0}{8}$$

to be trapped in one side of well

$$T_a = \frac{1}{2} m v_0^2 + \frac{2V_0}{9} - \frac{V_0}{8}$$

$$= \frac{1}{2} m v_0^2 + \frac{16V_0}{72} - \frac{9V_0}{72}$$

$$T_a = \frac{1}{2} m v_0^2 + \frac{7V_0}{72}$$

$$\alpha T_a > T_0$$

$$\alpha > \frac{T_0}{T_a}$$

$$\frac{1}{\alpha} < \frac{T_a}{T_0} = \frac{\frac{1}{2} m v_0^2 + \frac{7V_0}{72}}{\frac{1}{2} m v_0^2}$$

$$\frac{1}{\alpha} < 1 + \frac{7V_0}{36 m v_0^2}$$

$$\boxed{\alpha > \left[1 + \frac{7V_0}{36 m v_0^2}\right]^{-1}}$$

Turning points when $\alpha = 1$

$\alpha = 1 \Rightarrow 100\%$ loss of kinetic energy

$$E = V'(a) = \frac{V_0}{36}$$

$$\frac{V_0}{36} = V(a) + \frac{V_0}{4}$$

$$\frac{V_0}{36} - \frac{9V_0}{36} = \frac{-V_0 a^2 (a^2 + x^2)}{8a^4 + x^4}$$

$$\frac{8}{36} = \frac{a^2 (a^2 + x^2)}{8a^4 + x^4}$$

$$2\cdot(8a^4+x^4) = 9a^2(a^2+x^2)$$

$$2x^4+16a^4-9a^4-9a^2x^2=0$$

$$2x^4-9a^2x^2+7a^4=0 \qquad y=x^2$$

$$2y^2-9a^2y+7a^4=0$$

$$y=\frac{9a^2\pm\sqrt{81a^4-4(2)(7a^4)}}{4}$$

$$=\frac{9a^2\pm\sqrt{25a^4}}{4}$$

$$=\frac{9a^2\pm5a^2}{4}$$

$$=a^2 \text{ or } \frac{7}{2}a^2$$

$$x=\pm a \text{ or } \pm\left(\frac{7}{2}\right)^{1/2}a$$

$$\boxed{x=a,\left(\frac{7}{2}\right)^{1/2}a}$$

2-29 Derive Eq. 2.74 and 2.75, $x_0 = v_0 = 0$ at $t_0 = 0$

$$F = bv^2$$

$$m\frac{dv}{dt} = -mg + bv^2$$

$$\frac{m}{b}\frac{dv}{dt} = -\frac{mg}{b} + v^2$$

$$\frac{dv}{-\frac{mg}{b} + v^2} = \frac{b}{m}dt$$

$$\int_0^v \frac{dv}{-\frac{mg}{b} + v^2} = \frac{b}{m}t$$

$$-\int_0^v \frac{dv}{\frac{mg}{b} - v^2} = \frac{b}{m}t$$

$$-\sqrt{\frac{b}{mg}}\tanh^{-1}\left(v\sqrt{\frac{b}{mg}}\right) = \frac{b}{m}t$$

$$\tanh^{-1}\left(v\sqrt{\frac{b}{mg}}\right) = -\sqrt{\frac{bg}{m}}\,t$$

$$v = \sqrt{\frac{mg}{b}}\tanh\left(-\sqrt{\frac{bg}{m}}\,t\right)$$

$$\boxed{v = -\sqrt{\frac{mg}{b}}\tanh\left(\sqrt{\frac{bg}{m}}\,t\right) \text{ \{odd function\}}}$$

$$\frac{dx}{dt} = -\sqrt{\frac{mg}{b}}\tanh\left(\sqrt{\frac{bg}{m}}\,t\right)$$

$$x = \int_0^t -\sqrt{\frac{mg}{b}}\tanh\left(\sqrt{\frac{bg}{m}}\,t\right)dt$$

$$\boxed{x = -\frac{m}{b}\ln\left[\cosh\left(\sqrt{\frac{bg}{m}}\,t\right)\right] \quad \{\cosh x \geq 1\}}$$

Expansions for large and small t on next page.

$$1 - \tanh^2 y = \operatorname{sech}^2 y$$

$$v^2 = \frac{mg}{b}\tanh^2 y$$

$$v = \sqrt{\frac{mg}{b}}\tanh y$$

$$dv = \sqrt{\frac{mg}{b}}\operatorname{sech}^2 y\, dy$$

$$-\int \frac{dv}{\frac{mg}{b} - v^2} = -\sqrt{\frac{mg}{b}}\int \frac{\operatorname{sech}^2 y\, dy}{\frac{mg}{b} - \frac{mg}{b}\tanh^2 y}$$

$$= -\sqrt{\frac{mg}{b}}\frac{b}{mg}\int \frac{\operatorname{sech}^2 y\, dy}{1 - \tanh^2 y}$$

$$= -\sqrt{\frac{b}{mg}}\int \frac{\operatorname{sech}^2 y\, dy}{\operatorname{sech}^2 y\, dy} \qquad 1 - \tanh^2 y = \operatorname{sech}^2 y$$

$$= -\sqrt{\frac{b}{mg}}\, y + C$$

$$= -\sqrt{\frac{b}{mg}}\tanh^{-1}\left(v\sqrt{\frac{b}{mg}}\right) + C$$

$$-\sqrt{\frac{mg}{b}}\int \tanh\left(\sqrt{\frac{bg}{m}}\,t\right)dt = -\frac{m}{b}\int \tanh(u)\,du$$

$$u = \sqrt{\frac{bg}{m}}\,t$$

$$du = \sqrt{\frac{bg}{m}}\,dt$$

$$\sqrt{\frac{m}{bg}}\,du = dt$$

$$= -\frac{m}{b}\int \frac{\sinh u\, du}{\cosh u}$$

$$z = \cosh u$$

$$dz = \sinh u\, du$$

$$= -\frac{m}{b}\int \frac{du}{u}$$

$$= -\frac{m}{b}\ln|u| + C$$

$v = -\sqrt{\frac{mg}{b}} \tanh\left(\sqrt{\frac{bg}{m}}\, t\right)$

$\tanh z = z - \frac{z^3}{3} + \frac{2z^5}{15} - \dots$

$v \doteq -\sqrt{\frac{mg}{b}}\left(\sqrt{\frac{bg}{m}}\, t - \frac{\left(\frac{bg}{m}\right)^{3/2} t^3}{3} + \dots\right)$

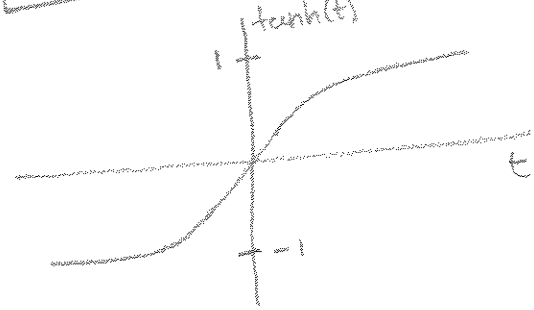
if $t \ll \sqrt{\frac{m}{bg}}$

$v \doteq -\sqrt{\frac{mg}{b}}\left(\sqrt{\frac{bg}{m}}\, t\right)$

$v \doteq -gt$

if $t \gg \sqrt{\frac{m}{bg}}$

$v \doteq -\sqrt{\frac{mg}{b}}\,(1) \quad \{ \tanh(t) \to 1$



$x = -\frac{m}{b} \ln \cosh\left(\sqrt{\frac{bg}{m}}\, t\right)$

$\cosh(z) = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots$

$\ln(x) = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \dots \quad |x| \leq 1$

if $t \ll \sqrt{\frac{m}{bg}}$

$x \doteq -\frac{m}{b} \ln\left(1 + \frac{bg}{2m} t^2\right)$

$x \doteq -\frac{m}{b}\left(\frac{bg}{2m} t^2\right)$

$x \doteq -\frac{1}{2} g t^2$

if $t \gg \sqrt{\frac{m}{bg}}$

$\cosh z = \frac{1}{2}\left(e^{z} + e^{-z}\right)$

$\cosh z \doteq \frac{1}{2} e^{z} \qquad z \gg 1$

$x \doteq -\frac{m}{b} \ln\left(\frac{e^{\sqrt{\frac{bg}{m}} t}}{2}\right)$

$\doteq -\frac{m}{b}\left[\sqrt{\frac{bg}{m}}\, t - \ln 2\right.$

$x \doteq \frac{m}{b} \ln 2 - \sqrt{\frac{mg}{b}}\, t$

2-31

i) Object starting at $x_0=0$ and reaching max height

$v(0)=v_0 \quad v_f=0 \quad t_0=0 \quad t_f=t_0$

$$m\frac{dv}{dt} = -mg - bv^2$$

$$-\frac{m}{b}\frac{dv}{dt} = \frac{mg}{b} + v^2$$

$$\frac{dv}{\frac{mg}{b}+v^2} = -\frac{b}{m}dt$$

$$\int_{v_0}^{0} \frac{dv}{\frac{mg}{b}+v^2} = \int_0^{t_0} -\frac{b}{m}dt$$

$$\sqrt{\frac{b}{mg}}\tan^{-1}\left(\sqrt{\frac{b}{mg}}v\right)\Bigg|_{v_0}^{0} = -\frac{b}{m}t\Bigg|_0^{t_0}$$

$$-\sqrt{\frac{b}{mg}}\tan^{-1}\left(\sqrt{\frac{b}{mg}}v_0\right) = -\frac{b}{m}t_0$$

$$\sqrt{\frac{m}{bg}}\tan^{-1}\left(\sqrt{\frac{b}{mg}}v_0\right) = t_0 \quad \{\text{time for max } x\}$$

$$\int \frac{dv}{\frac{mg}{b}+v^2} = \sqrt{\frac{mg}{b}}\int \frac{\sec^2\theta\, d\theta}{\frac{mg}{b}+\frac{mg}{b}\tan^2\theta} \qquad v^2 = \frac{mg}{b}\tan^2\theta$$

$$= \sqrt{\frac{b}{mg}}\int \frac{\sec^2\theta\, d\theta}{1+\tan^2\theta} \qquad v = \sqrt{\frac{mg}{b}}\tan\theta$$

$$dv = \sqrt{\frac{mg}{b}}\sec^2\theta\, d\theta$$

$$= \sqrt{\frac{b}{mg}}\,\theta + C$$

$$= \sqrt{\frac{b}{mg}}\tan^{-1}\left(\sqrt{\frac{b}{mg}}v\right) + C$$

$$\sqrt{\frac{m}{bg}}\tan^{-1}\left(\sqrt{\frac{b}{mg}}v\right) = t_0 - t \quad \{0 < t < t_0\}$$

$$v = \sqrt{\frac{mg}{b}}\tan\left(\sqrt{\frac{bg}{m}}(t_0-t)\right)$$

$$\int_0^x dx = \int_0^t \sqrt{\frac{mg}{b}}\tan\left(\sqrt{\frac{bg}{m}}(t_0-t)\right)dt$$

$$\sqrt{\frac{mg}{b}}\int \tan\left(\sqrt{\frac{bg}{m}}(t_0-t)\right)dt = -\frac{m}{b}\int \tan u\, du$$

$$u = \sqrt{\frac{bg}{m}}(t_0-t)$$

$$du = -\sqrt{\frac{bg}{m}}\,dt \qquad = \frac{m}{b}\ln\cos u + C$$

$$-\sqrt{\frac{m}{bg}}\,du = dt$$

$$x = \frac{m}{b}\ln\cos\left(\sqrt{\frac{bg}{m}}(t_0-t)\right)\Bigg|_0^t$$

$$= \frac{m}{b}\left[\ln\cos\left(\sqrt{\frac{bg}{m}}(t_0-t)\right) - \ln\cos\left(\sqrt{\frac{bg}{m}}t_0\right)\right]$$

$$\cos\left(\sqrt{\frac{bg}{m}}t_0\right) = \cos\left(\sqrt{\frac{bg}{m}}\sqrt{\frac{m}{bg}}\tan^{-1}\left(\sqrt{\frac{b}{mg}}v_0\right)\right)$$

$$= \cos\left(\tan^{-1}\left(\sqrt{\frac{b}{mg}}v_0\right)\right)$$

$$= \cos\theta$$

$$= \left(\sqrt{1+\frac{bv_0^2}{mg}}\right)^{-1}$$

$$\theta = \cos^{-1}\left(\frac{1}{\sqrt{1+\frac{b}{mg}v_0^2}}\right)$$

$$= \frac{m}{b}\left[-\ln\left(\sqrt{1+\frac{bv_0^2}{mg}}\right)^{-1} + \ln\cos\left(\sqrt{\frac{bg}{m}}(t_0-t)\right)\right]$$

$$\boxed{x = \frac{m}{2b}\left[\ln\left(1+\frac{bv_0^2}{mg}\right) + 2\ln\cos\left(\sqrt{\frac{bg}{m}}(t_0-t)\right)\right]}$$

ii): for $t > t_0$, $v_0 = 0$

$$m\frac{dv}{dt} = -mg + bv^2$$

$$\frac{m}{b}\frac{dv}{dt} = -\frac{mg}{b} + v^2$$

$$\frac{dv}{\frac{mg}{b} - v^2} = -\frac{b}{m}dt$$

$$\int_0^v \frac{dv}{\frac{mg}{b} - v^2} = \int_{t_0}^t -\frac{b}{m}dt$$

$$\int \frac{dv}{\frac{mg}{b} - v^2} = \frac{b}{mg}\sqrt{\frac{mg}{b}}\int \frac{\operatorname{sech}^2 u\, du}{1 + \tanh^2 u}$$

$$v^2 = \frac{mg}{b}\tanh^2 u \qquad = \sqrt{\frac{b}{mg}}\int du$$

$$v = \sqrt{\frac{mg}{b}}\tanh u \qquad = \sqrt{\frac{b}{mg}}\, u + C$$

$$dv = \sqrt{\frac{mg}{b}}\operatorname{sech}^2 u\, du \qquad = \sqrt{\frac{b}{mg}}\tanh^{-1}\left(\sqrt{\frac{b}{mg}}v\right) + C$$

$$\sqrt{\frac{b}{mg}}\tanh^{-1}\left(\sqrt{\frac{b}{mg}}v\right) = -\frac{b}{m}(t - t_0)$$

$$\tanh^{-1}\left(\sqrt{\frac{b}{mg}}v\right) = -\sqrt{\frac{bg}{m}}(t - t_0)$$

$$v = \sqrt{\frac{mg}{b}}\tanh\left(-\sqrt{\frac{bg}{m}}(t - t_0)\right)$$

$$= -\sqrt{\frac{mg}{b}}\tanh\left(\sqrt{\frac{bg}{m}}(t - t_0)\right) \quad \{\text{odd function}\}$$

$$\int_{x_h}^x dx = -\sqrt{\frac{mg}{b}}\int_{t_0}^t \tanh\left(\sqrt{\frac{bg}{m}}(t - t_0)\right)dt$$

$$\int \tanh\left(\sqrt{\frac{bg}{m}}(t - t_0)\right)dt = \sqrt{\frac{m}{bg}}\int \tanh u\, du$$

$$u = \sqrt{\frac{bg}{m}}(t - t_0)$$

$$du = \sqrt{\frac{bg}{m}}dt \qquad = \sqrt{\frac{m}{bg}}\int \frac{\sinh u\, du}{\cosh u}$$

$$\sqrt{\frac{m}{bg}}du = dt \qquad = \sqrt{\frac{m}{bg}}\ln\cosh u + C$$

$$x - x_h = -\frac{m}{b}\ln\cosh\left(\sqrt{\frac{bg}{m}}(t - t_0)\right)\Big|_{t_0}^t$$

$$x = x_h - \frac{m}{b}\ln\cosh\left(\sqrt{\frac{bg}{m}}(t - t_0)\right)$$

$$x_h = x\big|_{t=t_0}$$

$$= \frac{m}{2b}\ln\left(1 + \frac{bv_0^2}{mg}\right)$$

$$\boxed{x = \frac{m}{2b}\ln\left(1 + \frac{bv_0^2}{mg}\right) - \frac{m}{b}\ln\cosh\left(\sqrt{\frac{bg}{m}}(t - t_0)\right)}$$

2-33 $v_0 = v_e = \left(\frac{2MG}{x}\right)^{1/2}$

$$v = \frac{dx}{dt} = \left(\frac{2MG}{x}\right)^{1/2}$$

$$\frac{dx}{\left(\frac{2MG}{x}\right)^{1/2}} = dt$$

$$\int_{x_0}^{x} \frac{x^{1/2}\,dx}{(2MG)^{1/2}} = \int_0^t dt$$

$$\frac{1}{(2MG)^{1/2}} \frac{2}{3} x^{3/2} \Big|_{x_0}^{x} = t$$

$$\left(\frac{2}{9MG}\right)^{1/2} \left[x^{3/2} - x_0^{3/2}\right] = t$$

$$x^{3/2} - x_0^{3/2} = \left(\frac{9MG}{2}\right)^{1/2} t$$

$$x^{3/2} = x_0^{3/2} + \left(\frac{9MG}{2}\right)^{1/2} t$$

$$\boxed{x = \left[x_0^{3/2} + \left(\frac{9MG}{2}\right)^{1/2} t\right]^{2/3}}$$

$$\frac{1}{2}mv_0^2 + -\frac{mMG}{x} = E$$

$$= 0 \quad \{\text{Escape condition}\}$$

$$\frac{1}{2}mv_0^2 = \frac{mMG}{x}$$

$$v_0^2 = \frac{2MG}{x}$$

$$v_e = \sqrt{\frac{2MG}{x}}$$

2-35

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\cos^3\theta = \frac{1}{8}\left(e^{i\theta} + e^{-i\theta}\right)^3$$

$$= \frac{1}{8}\left(e^{i\theta} + e^{-i\theta}\right)\left(e^{i\theta} + e^{-i\theta}\right)\left(e^{i\theta} + e^{-i\theta}\right)$$

$$= \frac{1}{8}\left[e^{2i\theta} + 1 + 1 + e^{-2i\theta}\right]\left(e^{i\theta} + e^{-i\theta}\right)$$

$$= \frac{1}{8}\left[e^{3i\theta} + 2e^{i\theta} + e^{-i\theta} + e^{i\theta} + 2e^{-i\theta} + e^{-3i\theta}\right]$$

$$= \frac{1}{8}\left[e^{3i\theta} + e^{-3i\theta} + 3e^{i\theta} + 3e^{-i\theta}\right]$$

$$2\cos 3\theta = e^{3i\theta} + e^{-3i\theta}$$

$$6\cos\theta = 3e^{i\theta} + 3e^{-i\theta}$$

$$= \frac{1}{8}\left[2\cos 3\theta + 6\cos\theta\right]$$

$$\boxed{\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta}$$

2-37

$\omega_0^2 - \gamma^2$ is very small

$\omega_1 = (\omega_0^2 - \gamma^2)^{1/2}$

(2.133) $x = Ae^{-\gamma t}\cos(\omega_1 t + \theta)$

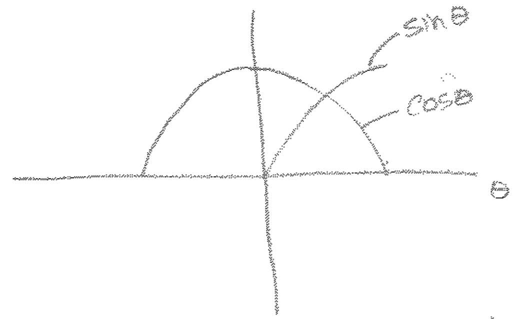
$x = Ae^{-\gamma t}\cos([\omega_0^2 - \gamma^2]^{1/2} t + \theta)$ $\quad \cos(A+B) = \cos A\cos B - \sin A \sin B$

$= Ae^{-\gamma t}[\cos\omega_1 t \cos\theta - \sin\omega_1 t \sin\theta]$

$= Ae^{-\gamma t}[\cos\theta - \omega_1 t \sin\theta]$

$= A\cos\theta\, e^{-\gamma t} - A\omega_1 t e^{-\gamma t}\sin\theta$

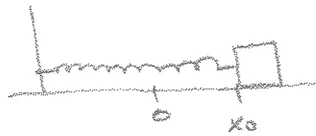
$$\boxed{x = (C_1 + C_2 t)e^{-\gamma t} \quad C_1 = A\cos\theta \quad C_2 = -A\omega_1\sin\theta}$$



θ small $\rightarrow \cos\theta \doteq 1$

$\sin\theta \simeq \theta$

2-39.



$t_0 = 0$
$v(0) = 0$
$x(0) = x_0$

i) underdamped, $\omega_0 > \gamma$

$$p = -\gamma \pm i\omega_1$$

$$x = C_1 e^{-\gamma t + i\omega_1 t} + C_2 e^{-\gamma t - i\omega_1 t}$$

$$= e^{-\gamma t}\left(C_1 e^{i\omega_1 t} + C_2 e^{-i\omega_1 t}\right)$$

$$= e^{-\gamma t}\left(B_1 \cos\omega_1 t + B_2 \sin\omega_1 t\right)$$

$$x_0 = x(0) = B_1$$

$$v = \dot{x} = -\gamma e^{-\gamma t}\left(B_1 \cos\omega_1 t + B_2 \sin\omega_1 t\right) + e^{-\gamma t}\left(-\omega_1 B_1 \sin\omega_1 t + \omega_1 B_2 \cos\omega_1 t\right)$$

$$0 = v(0) = -\gamma B_1 + \omega_1 B_2$$

$$B_2 = \frac{\gamma B_1}{\omega_1}$$

$$= \frac{x_0 \gamma}{\omega_1}$$

$$\boxed{x = e^{-\gamma t} x_0\left(\cos\omega_1 t + \frac{\gamma}{\omega_1}\sin\omega_1 t\right)}$$

ii) critically damped, $\omega_0 = \gamma$

$$x = (C_1 + C_2 t)e^{-\gamma t}$$

$$x_0 = x(0) = C_1$$

$$v = \dot{x} = -\gamma e^{-\gamma t}(C_1 + C_2 t) + e^{-\gamma t}(C_2)$$

$$0 = v(0) = -\gamma C_1 + C_2$$

$$C_2 = -\gamma C_1$$

$$C_2 = -\gamma x_0$$

$$x = (x_0 - \gamma x_0 t)e^{-\gamma t}$$

$$\boxed{x = x_0(1-\gamma t)e^{-\gamma t}}$$

iii) overdamped $\omega_0 < \gamma$

$$x = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t}$$

$$-\gamma_1 = -\gamma - (\gamma^2 - \omega_0^2)^{1/2}$$
$$-\gamma_2 = -\gamma + (\gamma^2 - \omega_0^2)^{1/2}$$

$$x_0 = x(0) = C_1 + C_2$$

$$v = \dot{x} = -\gamma_1 C_1 e^{-\gamma_1 t} - \gamma_2 C_2 e^{-\gamma_2 t}$$

$$0 = v(0) = -\gamma_1 C_1 - \gamma_2 C_2$$

$$C_1 = \frac{-\gamma_2}{\gamma_1} C_2$$

$$x_0 = \frac{-\gamma_2}{\gamma_1} C_2 + C_2$$
$$= \left(\frac{-\gamma_2}{\gamma_1} + 1\right) C_2$$
$$= \frac{-\gamma_2 + \gamma_1}{\gamma_1} C_2$$

$$\frac{x_0 \gamma_1}{-\gamma_2 + \gamma_1} = C_2$$

$$C_1 = \frac{-\gamma_2}{\gamma_1}\left(\frac{x_0 \gamma_1}{-\gamma_2 + \gamma_1}\right)$$

$$C_1 = \frac{-x_0 \gamma_2}{-\gamma_2 + \gamma_1}$$

$$x = \frac{-x_0 \gamma_2}{\gamma_1 - \gamma_2} e^{-\gamma_1 t} + \frac{x_0 \gamma_1}{\gamma_1 - \gamma_2} e^{-\gamma_2 t}$$

$$\boxed{x = \frac{x_0}{\gamma_1 - \gamma_2}\left(\gamma_1 e^{-\gamma_2 t} - \gamma_2 e^{-\gamma_1 t}\right)}$$

2-4. $X(0) = X_0 \quad V(0) = -V_0$

i) underdamped, $\omega_0 > \gamma$

$$p = -\gamma \pm i\omega_1$$

$$X = e^{-\gamma t}(B_1 \cos\omega_1 t + B_2 \sin\omega_1 t)$$

$$X_0 = X(0) = B_1$$

$$-V_0 = V(0) = -\gamma B_1 + \omega_1 B_2$$

$$-V_0 = -\gamma X_0 + \omega_1 B_2$$

$$\frac{\gamma X_0 - V_0}{\omega_1} = B_2$$

$$\boxed{X = e^{-\gamma t}\left(X_0 \cos\omega_1 t + \frac{\gamma X_0 - V_0}{\omega_1}\sin\omega_1 t\right)}$$

ii) critically damped, $\omega_0 = \gamma$

$$X = (C_1 + C_2 t)e^{-\gamma t}$$

$$X_0 = X(0) = C_1$$

$$-V_0 = V(0) = -\gamma C_1 + C_2$$

$$-V_0 = -X_0\gamma + C_2$$

$$X_0\gamma - V_0 = C$$

$$\boxed{X = (X_0 + (X_0\gamma - V_0)t)e^{-\gamma t}}$$

iii) overdamped, $\omega_0 < \gamma$

$$X = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t}$$

$$X_0 = X(0) = C_1 + C_2$$

$$-V_0 = V(0) = -\gamma_1 C_1 - \gamma_2 C_2$$

$$\gamma_2 C_2 - V_0 = -\gamma_1 C_1$$

$$\frac{V_0 - \gamma_2 C_2}{\gamma_1} = C_1$$

$$C_1 + C_2 = x_0$$

$$\frac{v_0 - \gamma_2 C_2}{\gamma_1} + C_2 = x_0$$

$$\frac{v_0}{\gamma_1} - \frac{\gamma_2 C_2}{\gamma_1} + C_2 = x_0$$

$$C_2\left(1 - \frac{\gamma_2}{\gamma_1}\right) = x_0 - \frac{v_0}{\gamma_1}$$

$$C_2\left(\frac{\gamma_1 - \gamma_2}{\gamma_1}\right) = \frac{x_0\gamma_1 - v_0}{\gamma_1}$$

$$C_2 = \frac{x_0\gamma_1 - v_0}{\gamma_1 - \gamma_2}$$

$$C_1 + \frac{x_0\gamma_1 - v_0}{\gamma_1 - \gamma_2} = x_0$$

$$C_1 = \frac{x_0(\gamma_1 - \gamma_2) - x_0\gamma_1 + v_0}{\gamma_1 - \gamma_2}$$

$$= \frac{x_0\gamma_1 - x_0\gamma_2 - x_0\gamma_1 + v_0}{\gamma_1 - \gamma_2}$$

$$= \frac{v_0 - x_0\gamma_2}{\gamma_1 - \gamma_2}$$

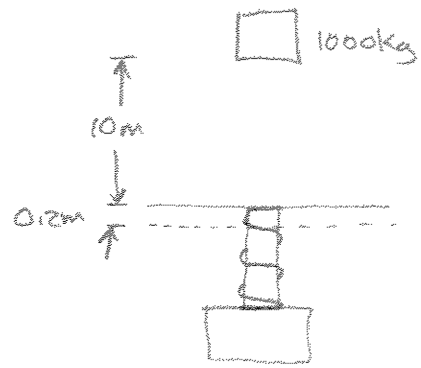
$$x = \frac{v_0 - x_0\gamma_2}{\gamma_1 - \gamma_2} e^{-\gamma_1 t} + \frac{x_0\gamma_1 - v_0}{\gamma_1 - \gamma_2} e^{-\gamma_2 t}$$

$$\boxed{\begin{aligned} x &= (\gamma_1 - \gamma_2)^{-1}\left([v_0 - x_0\gamma_2]e^{-\gamma_1 t} + [x_0\gamma_1 - v_0]e^{-\gamma_2 t}\right) \quad (v_0 > 0) \\ x &= (\gamma_1 - \gamma_2)^{-1}\left([x_0\gamma_1 + v_0]e^{-\gamma_2 t} - [x_0\gamma_2 + v_0]e^{-\gamma_1 t}\right) \quad (v_0 < 0) \end{aligned}}$$

2-43

critically damped

$$x = \left(x_0 + (x_0\gamma - v_0)t\right)e^{-\gamma t}$$



$$x = (0.2 + 0t)\,e^{-70t}$$

$$\boxed{x = 0.2\,e^{-70t}}$$

$$.001 = 0.2\,e^{-70t}$$

$$\frac{.001}{0.2} = e^{-70t}$$

$$.005 = e^{-70t}$$

$$\ln(.005) = -70t$$

$$\boxed{.076\,s = t}$$

v_0: $\frac{1}{2}mv^2 = mgh$

$$v^2 = 2gh$$

$$v = \sqrt{2gh}$$

$$= 14\,m/s$$

$$kx = mg$$

$$k = \frac{mg}{x}$$

$$= \frac{(1000\,kg)(9.81\,m/s^2)}{0.2\,m}$$

$$\boxed{k = 4.9 \times 10^4 \frac{kg}{s^2}}$$

To avoid overshoot:

$$x_0\gamma - v_0 = 0$$

$$\gamma = \frac{v_0}{x_0}$$

$$= \frac{14\,m/s}{0.2\,m}$$

$$\boxed{\gamma = 70\,s^{-1}}$$

$$\frac{b}{2m} = \gamma$$

$$b = \gamma\, 2m$$

$$\boxed{b = 1.4 \times 10^5 \frac{kg}{s}}$$

2-45

a) $m\ddot{x}+b\dot{x}+Kx=F_o$

x_h: $\ddot{x}+\frac{b}{m}\dot{x}+\frac{K}{m}x=0$ $\qquad x=e^{pt}$

$p^2+\frac{b}{m}p+\frac{K}{m}=0$

$p=-\frac{b}{2m}\pm\sqrt{\left(\frac{b}{2m}\right)^2-\frac{K}{m}}$ $\qquad \gamma=\frac{b}{2m} \quad \omega_0=\sqrt{\frac{K}{m}}$

$\omega_1^2=\omega_0^2-\gamma^2$

$p=-\gamma\pm i\omega_1$

$x_h=C_1e^{-\gamma t+i\omega_1 t}+C_2e^{-\gamma t-i\omega_1 t}$

$=e^{-\gamma t}\left[C_1e^{i\omega_1 t}+C_2e^{-i\omega_1 t}\right]$ $\qquad C_2=C_1^*=\frac{1}{2}Ae^{-i\theta}$

$=e^{-\gamma t}\left[\frac{1}{2}Ae^{i(\omega_1+\theta)t}+\frac{1}{2}Ae^{-i(\omega_1+\theta)t}\right]$

$x_h=Ae^{-\gamma t}\cos(\omega_1 t+\theta)$ $\qquad \left\{\cos\alpha=\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right\}$

$x_p=C$

$\dot{x}_p=\ddot{x}_p=0$

$KC=F_o$

$C=\frac{F_o}{K}$

$x_p=\frac{F_o}{K}$

$x=x_h+x_p$

$$\boxed{x=Ae^{-\gamma t}\cos(\omega_1 t+\theta)+\frac{F_o}{K}}$$

2-47

$$m\ddot{x} + Kx = F_0 \sin\omega t \ , \ v(0)=0 \ , \ x(0)=0$$

x_h: $\ddot{x} + \frac{K}{m}x = 0$

$$p^2 + \frac{K}{m} = 0 \qquad x = e^{pt}$$

$$p^2 = -\frac{K}{m}$$

$$p = \pm i\sqrt{\frac{K}{m}}$$

$$p = \pm i\omega_0 \qquad \omega_0 = \sqrt{\frac{K}{m}}$$

$$x_h = C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} \qquad C_1 = \frac{A}{2}e^{i\theta} \ C_2 = C_1^* = \frac{A}{2}e^{-i\theta}$$

$$= \frac{A}{2}e^{i(\omega_0 t+\theta)} + \frac{A}{2}e^{-i(\omega_0 t+\theta)}$$

$$x_h = A\cos(\omega_0 t + \theta)$$

$$x_p = C_3 \sin(\omega t + \theta_1)$$

$$\dot{x}_p = \omega C_3 \cos(\omega t + \theta_1)$$

$$\ddot{x}_p = -\omega^2 C_3 \sin(\omega t + \theta_1)$$

$$-\omega^2 C_3 \sin(\omega t + \theta_1) + \frac{K}{m}C_3 \sin(\omega t + \theta_1) = \frac{F_0}{m}\sin\omega t$$

$$C_3\left[\frac{K}{m} - \omega^2\right]\sin(\omega t + \theta_1) = F_0 \sin\omega t$$

$$C_3\left[\frac{K - m\omega^2}{m}\right] = \frac{F_0}{m} \qquad \sin(\omega t + \theta_1) = \sin\omega t$$

$$C_3 = \frac{F_0}{K - m\omega^2} \qquad \theta_1 = 0$$

$$x_p = \frac{F_0}{K - m\omega^2}\sin\omega t$$

$$x(t) = x_h(t) + x_p(t)$$

$$= A\cos(\omega_0 t + \theta) + \frac{F_0}{K - m\omega^2}\sin\omega t$$

$$0 = x(0) = A\cos\theta$$

$$v(t) = \dot{x}(t) = -A\omega_0 \sin(\omega_0 t + \theta) + \frac{F_0\omega}{K - m\omega^2}\cos\omega t$$

$$0 = v(0) = -A\omega_0 \sin\theta + \frac{F_0\omega}{K - m\omega^2}$$

$$A\cos\theta = 0$$

$$\theta = \frac{\pi}{2}$$

$$A\omega_0 \sin\theta = \frac{F_0 \omega}{k - m\omega^2}$$

$$A = \frac{F_0 \omega}{\omega_0 (k - m\omega^2)}$$

$$x(t) = \frac{F_0 \omega}{\omega_0 (k - m\omega^2)} \cos\left(\omega_0 t + \frac{\pi}{2}\right) + \frac{F_0}{k - m\omega^2} \sin\omega t$$

$$= \frac{-F_0 \omega}{\omega_0 (k - m\omega^2)} \sin\omega_0 t + \frac{F_0}{k - m\omega^2} \sin\omega t$$

$$= \frac{F_0}{\omega_0 (k - m\omega^2)} \left[\omega_0 \sin\omega t - \omega \sin\omega_0 t\right]$$

$$\boxed{x(t) = \frac{F_0}{m\omega_0 (\omega_0^2 - \omega^2)} \left[\omega_0 \sin\omega t - \omega \sin\omega_0 t\right]}$$

2-49

$$\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t, \quad x(0) = x_0, \quad v(0) = v_0, \quad \omega_0 = \gamma$$

$$\ddot{x} + 2\omega_0\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

x_h:

$$\ddot{x} + 2\omega_0\dot{x} + \omega_0^2 x = 0 \qquad x = e^{pt}$$

$$p^2 + 2\omega_0 p + \omega_0^2 = 0$$

$$p = -\frac{2\omega_0}{2} \pm \frac{\sqrt{4\omega_0^2 - 4\omega_0^2}}{2}$$

$$= -\omega_0 = \gamma$$

$$x_h = (C_1 + C_2 t)e^{-\gamma t}$$

x_p:

$$x_p = A\cos\omega t + B\sin\omega t$$

$$\dot{x} = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$\ddot{x} = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

$$\ddot{x} + 2\omega_0\dot{x} + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

$$-\omega^2 A\cos\omega t - \omega^2 B\sin\omega t + 2\omega_0(-\omega A\sin\omega t + \omega B\cos\omega t) + \omega_0^2(A\cos\omega t + B\sin\omega t) = \frac{F_0}{m}\cos\omega t$$

$$-\omega^2 A\cos\omega t + 2\omega_0\omega B\cos\omega t + \omega_0^2 A\cos\omega t = \frac{F_0}{m}\cos\omega t$$

$$-\omega^2 A + 2\omega_0\omega B + \omega_0^2 A = \frac{F_0}{m}$$

$$-\omega^2 B\sin\omega t - 2\omega_0\omega A\sin\omega t + \omega_0^2 B\sin\omega t = 0$$

$$-\omega^2 B - 2\omega_0\omega A + \omega_0^2 B = 0$$

$$A(\omega_0^2 - \omega^2) + 2\omega_0\omega B = \frac{F_0}{m} \qquad B(\omega_0^2 - \omega^2) - 2\omega_0\omega A = 0$$

$$B = \frac{2\omega_0\omega A}{(\omega_0^2 - \omega^2)}$$

$$A(\omega_0^2 - \omega^2) + \frac{(2\omega_0\omega)^2 A}{(\omega_0^2 - \omega^2)} = \frac{F_0}{m}$$

$$A\left[\frac{(\omega_0^2-\omega^2)^2+4\omega_0^2\omega^2}{\omega_0^2-\omega^2}\right]=\frac{F_0}{m}$$

$$A=\frac{F_0}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2-\omega^2)^2+4\omega_0^2\omega^2}$$

$$=\frac{F_0}{m}\frac{\omega_0^2-\omega^2}{\omega_0^4-2\omega_0^2\omega^2+\omega^4+4\omega_0^2\omega^2}$$

$$=\frac{F_0}{m}\frac{\omega_0^2-\omega^2}{\omega_0^4+2\omega_0^2\omega^2+\omega^4}$$

$$=\frac{F_0}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2+\omega^2)^2}$$

$$B=\frac{2\omega_0\omega}{\omega_0^2-\omega^2}\frac{F_0}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2+\omega^2)^2}$$

$$=\frac{F_0}{m}\frac{2\omega_0\omega}{(\omega_0^2+\omega^2)^2}$$

$$x_p=\frac{F_0}{m}\frac{1}{(\omega_0^2+\omega^2)^2}\left[(\omega_0^2-\omega^2)\cos\omega t+2\omega_0\omega\sin\omega t\right]$$

$$x(t)=x_h(t)+x_p(t)$$

$$=(C_1+C_2t)e^{-\gamma t}+\frac{F_0}{m}\frac{1}{(\omega_0^2+\omega^2)^2}\left[(\omega_0^2-\omega^2)\cos\omega t+2\gamma\omega\sin\omega t\right]$$

$$x_0=x(0)=C_1+\frac{F_0}{m}\frac{1}{(\omega_0^2+\omega^2)^2}\left[(\omega_0^2-\omega^2)\right]$$

$$C_1=x_0-\frac{F_0}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2+\omega^2)^2}$$

$$\dot{x}=-\gamma(C_1+C_2t)e^{-\gamma t}+C_2e^{-\gamma t}+\frac{F_0}{m}\frac{1}{(\omega_0^2+\omega^2)^2}\left[-\omega(\omega_0^2-\omega^2)\sin\omega t+2\gamma\omega^2\cos\omega t\right]$$

$$v_0=v(0)=-\gamma C_1+C_2+\frac{F_0}{m}\frac{1}{(\omega_0^2+\omega^2)^2}\left[2\gamma\omega^2\right]$$

$$v_0=-\gamma\left[x_0-\frac{F_0}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2+\omega^2)^2}\right]+C_2+\frac{F_0}{m}\frac{2\gamma\omega^2}{(\omega_0^2+\omega^2)^2}$$

$$C_2 = V_0 + \gamma\left[X_0 - \frac{F_0}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2+\omega^2)^2}\right] - \frac{F_0}{m}\frac{2\gamma\omega^2}{(\omega_0^2+\omega^2)^2}$$

$$= V_0 + \gamma X_0 - \gamma\frac{F_0}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2+\omega^2)^2} - \frac{F_0}{m}\frac{2\gamma\omega^2}{(\omega_0^2+\omega^2)^2}$$

$$= V_0 + \gamma X_0 - \frac{F_0}{m}\frac{1}{(\omega_0^2+\omega^2)^2}\left[\gamma(\omega_0^2-\omega^2) - 2\gamma\omega^2\right]$$

$$X(t) = \left(X_0 - \frac{F_0}{m}\frac{\omega_0^2-\omega^2}{(\omega_0^2+\omega^2)^2} + \left[V_0+\gamma X_0 - \frac{F_0}{m}\frac{1}{(\omega_0^2+\omega^2)^2}\left[\gamma\omega_0^2 - 3\gamma\omega^2\right]\right]t\right)e^{-\gamma t} +$$

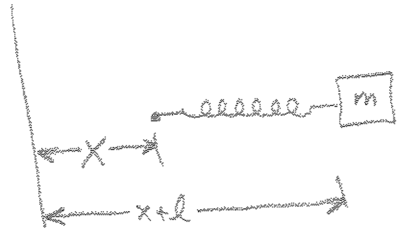
$$\frac{F_0}{m}\frac{1}{(\omega_0^2+\omega^2)^2}\left[(\omega_0^2-\omega^2)\cos\omega t + 2\gamma\omega\sin\omega t\right]$$

$$= \frac{F_0}{m(\omega_0^2+\omega^2)^2}\left[\left(\omega^2-\omega_0^2 + \left[-\gamma\omega_0^2 + 3\gamma\omega^2\right]t\right)e^{-\gamma t} + (\omega_0^2-\omega^2)\cos\omega t + 2\gamma\omega\sin\omega t\right]$$

$$+ \left(X_0 + \left[V_0 + \gamma X_0\right]t\right)e^{-\gamma t}$$

$$\boxed{X(t) = \frac{F_0}{m(\gamma^2+\omega^2)^2}\left[(\gamma^2-\omega^2)\cos\omega t + 2\gamma\omega\sin\omega t - \left(\gamma^2-\omega^2+\gamma^3 t - 3\gamma\omega^2 t\right)e^{-\gamma t}\right] + \left(X_0 + V_0 t + \gamma X_0 t\right)e^{-\gamma t}}$$

2-51



l = relaxed length

$X = a\sin\omega t$

$m\ddot{x} = -b\dot{x} - kx'$ $\qquad x' = x + l - X$

$m\ddot{x} = -b\dot{x} - k(x + l - X)$

$= -b\dot{x} - kx - kl + kX$

$= -b\dot{x} - kx - kl + ka\sin\omega t$

$m\ddot{x} + b\dot{x} + kx = -kl + ka\sin\omega t$ $\qquad kl = 0$

$$\boxed{m\ddot{x} + b\dot{x} + kx = ka\sin\omega t}$$

If $f = -b(\dot{x} - \dot{X})$ then

$f = -b\dot{x} + \omega ba\cos\omega t$

So,

$m\ddot{x} = -b(\dot{x} - \dot{X}) + ka\sin\omega t - kx - kl$

$= -b\dot{x} + \omega ba\cos\omega t + ka\sin\omega t - kx$

$$\boxed{m\ddot{x} + b\dot{x} + kx = \omega ba\cos\omega t + ka\sin\omega t}$$

2-53

$0 < t < \frac{3\pi}{2\omega_0}$ $\quad m\ddot{x} + kx = 0 \quad x(0) = 0,\ V(0) = V_0$

$\ddot{x} + \frac{k}{m} x = 0$

$\ddot{x} + \omega_0^2 x = 0 \quad x = e^{pt}$

$p^2 + \omega_0^2 = 0$

$p = \pm j\omega_0$

$x = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$

$0 = x(0) = C_1$

$\dot{x} = -\omega_0 C_1 \sin\omega_0 t + \omega_0 C_2 \cos\omega_0 t$

$V_0 = V(0) = \omega_0 C_2$

$C_2 = \frac{V_0}{\omega_0}$

$$\boxed{x = \frac{V_0}{\omega_0} \sin\omega_0 t, \quad 0 < t < \frac{3\pi}{2\omega_0}}$$

$t \geq \frac{3\pi}{2\omega_0}$ $\quad m\ddot{x} + kx = B\cos(\omega t + \theta)$

$\ddot{x} + \omega_0^2 x = \frac{B}{m}\cos(\omega t + \theta)$

initial conditions

$x\left(\frac{3\pi}{2\omega_0}\right) = \frac{V_0}{\omega_0}\sin\left(\frac{3\pi}{2}\right)$

$= \frac{-V_0}{\omega_0}$

$\dot{x} = V_0 \cos\omega_0 t$

$\dot{x}\left(\frac{3\pi}{2\omega_0}\right) = 0$

$\therefore x_h = K_1 \cos\omega_0 t + K_2 \sin\omega_0 t$

$x_p = C_3 \cos(\omega t + \theta) + C_4 \sin(\omega t + \theta)$

$\dot{x}_p = -\omega C_3 \sin(\omega t + \theta) + \omega C_4 \cos(\omega t + \theta)$

$\ddot{x}_p = -\omega^2 C_3 \cos(\omega t + \theta) - \omega^2 C_4 \sin(\omega t + \theta)$

$-\omega^2 C_3 \cos(\omega t + \theta) - \omega^2 C_4 \sin(\omega t + \theta) + \omega_0^2 C_3 \cos(\omega t + \theta) + \omega_0^2 C_4 \sin(\omega t + \theta) = \frac{B}{m}\cos(\omega t + \theta)$

$C_3(\omega_0^2 - \omega^2) = \frac{B}{m} \qquad C_4(\omega_0^2 - \omega^2) = 0$

$C_3 = \frac{B}{m}(\omega_0^2 - \omega^2)^{-1} \qquad C_4 = 0$

$$x = x_h + x_p$$

$$x = K_1 \cos\omega_0 t + K_2 \sin\omega_0 t + \frac{B}{m}(\omega_0^2-\omega^2)^{-1}\cos(\omega t+\theta)$$

$$\frac{-V_0}{\omega_0} = x\left(\frac{3\pi}{2\omega_0}\right) = -K_2 + \frac{B}{m}(\omega_0^2-\omega^2)^{-1}\cos\alpha \qquad \alpha = \frac{3\pi\omega}{2\omega_0}+\theta$$

$$K_2 = \frac{V_0}{\omega_0} + \frac{B}{m}(\omega_0^2-\omega^2)^{-1}\cos\alpha$$

$$\dot{x} = -\omega_0 K_1 \sin\omega_0 t + \omega_0 K_2 \cos\omega_0 t - \frac{B\omega}{m}(\omega_0^2-\omega^2)^{-1}\sin(\omega t+\theta)$$

$$0 = \dot{x}\left(\frac{3\pi}{2\omega_0}\right) = \omega_0 K_1 - \frac{B\omega}{m}(\omega_0^2-\omega^2)\sin\alpha$$

$$K_1 = \frac{B\omega}{m\omega_0}(\omega_0^2-\omega^2)\sin\alpha$$

$$x = \frac{B}{m}(\omega_0^2-\omega^2)^{-1}\frac{\omega}{\omega_0}\sin\alpha\cos\omega_0 t + \left[\frac{V_0}{\omega_0} + \frac{B}{m}(\omega_0^2-\omega^2)^{-1}\cos\alpha\right]\sin\omega_0 t + \frac{B}{m}(\omega_0^2-\omega^2)^{-1}\cos(\omega t+\theta)$$

$$\boxed{x = \frac{B}{m}(\omega_0^2-\omega^2)\left[\frac{\omega}{\omega_0}\sin\alpha\cos\omega_0 t + \cos\alpha\sin\omega_0 t + \cos(\omega t+\theta)\right] + \frac{V_0}{\omega_0}\sin\omega_0 t, \quad t \geq \frac{3\pi}{2\omega_0} \qquad \alpha = \frac{3\pi\omega}{2\omega_0}+\theta}$$

2-55

$$\ddot{x} + \frac{k}{m}x = F(t) \qquad F(t) = \begin{cases} 0, & t < t_0 \\ P_0/\delta t, & t_0 \le t \le t_0 + \delta t \\ 0, & t > t_0 + \delta t \end{cases}$$

$x(0) = 0$
$v(0) = 0$

a) i) $t < t_0$

$$\ddot{x} + \omega_0^2 x = 0$$

$$x = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t$$

$$0 = x(0) = C_1$$

$$\dot{x} = -\omega_0 C_1 \sin\omega_0 t + \omega_0 C_2 \cos\omega_0 t$$

$$0 = v(0) = C_2$$

$$\boxed{x = 0 \quad t < t_0}$$

ii) $t_0 \le t \le t_0 + \delta t$

$$\ddot{x} + \omega_0^2 x = \frac{P_0}{m\delta t} \qquad x(t_0) = 0, \quad v(t_0) = 0$$

$$x_h = C_1 \cos\omega_0(t - t_0) + C_2 \sin\omega_0(t - t_0)$$

$$x_p = K_1$$

$$\frac{k}{m}K_1 = \frac{P_0}{\delta t m}$$

$$K_1 = \frac{m}{k}\frac{P_0}{\delta t}$$

$$x_p = \frac{P_0}{k\delta t}$$

$$x = x_h + x_p$$

$$= C_1 \cos\omega_0(t - t_0) + C_2 \sin\omega_0(t - t_0) + \frac{P_0}{k\delta t}$$

$$0 = x(t_0) = C_1 \cos(0) + C_2 \sin(0) + \frac{P_0}{k\delta t}$$

$$C_1 = -\frac{P_0}{k\delta t}$$

$$\dot{x} = -\omega_0 C_1 \sin\omega_0(t - t_0) + \omega_0 C_2 \cos\omega_0(t - t_0)$$

$$0 = \dot{x}(t_0) = \omega_0 C_2$$

$$C_2 = 0$$

$$X = \frac{-P_0}{K\delta t}\cos\omega_0(t-t_0) + \frac{P_0}{K\delta t}$$

$$\boxed{X = \frac{P_0}{K\delta t}\left[1-\cos\omega_0(t-t_0)\right], \quad t_0 \le t \le t_0+\delta t}$$

iii) $t > t_0 + \delta t$

$$\ddot{x} + \omega_0^2 x = 0$$

$$x = C_1\cos\omega_0(t-(t_0+\delta t)) + C_2\sin\omega_0(t-(t_0+\delta t))$$

initial conditions:

$$x(t_0+\delta t) = \frac{P_0}{K\delta t}\left[1-\cos\omega_0(\delta t)\right]$$

$$\dot{x} = \frac{P_0\omega_0}{K\delta t}\sin\omega_0(t-t_0)$$

$$\dot{x}(t_0+\delta t) = \frac{P_0\omega_0}{K\delta t}\sin\omega_0(\delta t)$$

$$x(t_0+\delta t) = C_1\cos(0) + C_2\sin(0)$$

$$\frac{P_0}{K\delta t}\left[1-\cos\omega_0(\delta t)\right] = C_1$$

$$\dot{x} = -C_1\omega_0\sin\omega_0(t-(t_0+\delta t)) + C_2\omega_0\cos\omega_0(t-(t_0+\delta t))$$

$$\dot{x}(t_0+\delta t) = C_2\omega_0$$

$$\sin\omega_0(\delta t)\frac{P_0\omega_0}{K\delta t} = C_2\omega_0$$

$$C_2 = \frac{P_0}{K\delta t}\sin\omega_0(\delta t)$$

$$x = \frac{P_0}{K\delta t}\left[(1-\cos\omega_0(\delta t))\cos\omega_0(t-(t_0+\delta t)) + \sin\omega_0(\delta t)\sin\omega_0(t-(t_0+\delta t))\right]$$

$$= \frac{P_0}{K\delta t}\left[(1-\cos\omega_0(\delta t))\cos\omega_0([t-t_0]-\delta t) + \sin\omega_0(\delta t)\sin\omega_0([t-t_0]-\delta t)\right]$$

$$\cos\omega_0([t-t_0]-\delta t) = \cos\omega_0(t-t_0)\cos\omega_0\delta t + \sin\omega_0(t-t_0)\sin\omega_0\delta t$$

$$\sin\omega_0([t-t_0]-\delta t) = \sin\omega_0(t-t_0)\cos\omega_0\delta t - \cos\omega_0(t-t_0)\sin\omega_0\delta t$$

$$-\cos\omega_0\delta t\cos\omega_0([t-t_0]-\delta t) = -\cos^2\omega_0\delta t\cos\omega_0(t-t_0) - \sin\omega_0\delta t\cos\omega_0\delta t\sin\omega_0(t-t_0)$$

$$\sin\omega_0\delta t\sin\omega_0([t-t_0]-\delta t) = \sin\omega_0\delta t\cos\omega_0\delta t\sin\omega_0(t-t_0) - \sin^2\omega_0\delta t\cos\omega_0(t-t_0)$$

$$-\cos\omega_0\delta t\cos\omega_0([t-t_0]-\delta t + \sin\omega_0\delta t\sin\omega_0([t-t_0-\delta t) =$$

$$-\cos\omega_0(t-t_0)(\cos^2\omega_0\delta t + \sin^2\omega_0\delta t) = -\cos\omega_0(t-t_0)$$

$$\boxed{x = \frac{P_0}{k\delta t}\left[\cos\omega_0([t-t_0]-\delta t) - \cos\omega_0(t-t_0)\right], \quad t > t_0 + \delta t}$$

2-57 $m\ddot{x} + b\dot{x} + kx = F$, $\gamma = \frac{\omega_0}{3}$, $x(0)=0$, $v(0)=0$, $F = A\sin\omega_0 t + B\sin 3\omega_0 t$

a) $m\ddot{x}_n + b\dot{x}_n + kx_n = F_n(t)$

$F_1 = A\sin\omega_0 t$

$x_h = 0 \quad (x(0)=0, v(0)=0)$

$x_{p_1} = K_{11}\cos\omega_0 t + K_{21}\sin\omega_0 t$

$\dot{x}_{p_1} = -\omega_0 K_{11}\sin\omega_0 t + \omega_0 K_{21}\cos\omega_0 t$

$\ddot{x}_{p_1} = -\omega_0^2 K_{11}\cos\omega_0 t - \omega_0^2 K_{21}\sin\omega_0 t$

$\ddot{x}_{p_1} + \frac{b}{m}\dot{x}_{p_1} + \omega_0^2 x_{p_1} = \frac{A\sin\omega_0 t}{m}$

$\cos\omega_0 t$: $-\omega_0^2 K_{11} + \frac{b}{m}\omega_0 K_{21} + K_{11}\omega_0^2 = 0$

$\sin\omega_0 t$: $-\omega_0^2 K_{21} - \frac{b}{m}\omega_0 K_{11} + K_{21}\omega_0^2 = \frac{A}{m}$

$\frac{b}{m} = 2\gamma$
$= \frac{2\omega_0}{3}$

$\cos\omega_0 t$: $-\omega_0^2 K_{11} + \frac{2\omega_0^2}{3}K_{21} + \omega_0^2 K_{11} = 0$

$\frac{2\omega_0^2}{3}K_{21} = 0$

$K_{21} = 0$

$\sin\omega_0 t$: $-\omega_0^2 K_{21} - \frac{2\omega_0^2}{3}K_{11} + \omega_0^2 K_{21} = \frac{A}{m}$

$-\frac{2\omega_0^2}{3}K_{11} = \frac{A}{m}$

$K_{11} = -\frac{3A}{2m\omega_0^2}$

$x_{p_1} = \frac{-3A}{2m\omega_0^2}\cos\omega_0 t$

$F_2 = B\sin 3\omega_0 t$

$x_h = 0 \quad (x(0)=0, v(0)=0)$

$x_{p_2} = K_{12}\cos 3\omega_0 t + K_{22}\sin 3\omega_0 t$

$\dot{x}_{p_2} = -3\omega_0 K_{12}\sin 3\omega_0 t + 3\omega_0 K_{22}\cos 3\omega_0 t$

$\ddot{x}_{p_2} = -9\omega_0^2 K_{12}\cos 3\omega_0 t - 9\omega_0^2 K_{22}\sin 3\omega_0 t$

$\ddot{x}_{p_2} + \frac{b}{m}\dot{x}_{p_2} + \omega_0^2 x_{p_2} = \frac{B}{m}\sin 3\omega_0 t$

$\cos 3\omega_0 t$: $-9\omega_0^2 K_{12} + \frac{b}{m}3\omega_0 K_{22} + \omega_0^2 K_{12} = 0$

$-9\omega_0^2 K_{12} + 2\omega_0^2 K_{22} + \omega_0^2 K_{12} = 0$

$-8\omega_0^2 K_{12} + 2\omega_0^2 K_{22} = 0$

$\sin 3\omega_0 t$: $-9\omega_0^2 K_{22} - 2\omega_0^2 K_{12} + \omega_0^2 K_{22} = \frac{B}{m}$

$-8\omega_0^2 K_{22} - 2\omega_0^2 K_{12} = \frac{B}{m}$

$-8\omega_0^2 K_{12} + 2\omega_0^2 K_{22} = 0$

$-2\omega_0^2 K_{12} - 8\omega_0^2 K_{22} = \frac{B}{m}$

$-8\omega_0^2 K_{12} + 2\omega_0^2 K_{22} = 0$

$\underline{8\omega_0^2 K_{12} + 32\omega_0^2 K_{22} = -4B/m}$

$34\omega_0^2 K_{22} = -4B/m$

$K_{22} = \frac{-2B}{17\omega_0^2 m}$

$-8\omega_0^2 K_{12} + 2\omega_0^2\left(\frac{-2B}{17\omega_0^2 m}\right) = 0$

$-8\omega_0^2 K_{12} - \frac{4B}{17m} = 0$

$-8\omega_0^2 K_{12} = \frac{4B}{17m}$

$K_{12} = \frac{-B}{34m\omega_0^2}$

$$x_{P_1} = \frac{-3A}{2m\omega_0^2}\cos\omega_0 t \qquad x_{P_2} = \frac{-B}{34m\omega_0^2}\cos 3\omega_0 t - \frac{2B}{17m\omega_0^2}\sin 3\omega_0 t$$

$$= \frac{-B}{34m\omega_0^2}\left[\cos 3\omega_0 t - 4\sin 3\omega_0 t\right]$$

$$x_p(t) = \sum_n^4 x_{p_n}(t)$$

$$= \frac{-3A}{2m\omega_0^2}\cos\omega_0 t - \frac{B}{34m\omega_0^2}\left[\cos 3\omega_0 t - 4\sin 3\omega_0 t\right]$$

$$= \frac{1}{m\omega_0^2}\left[-\frac{3A}{2}\cos\omega_0 t - \frac{B}{34}\left(\cos 3\omega_0 t - 4\sin 3\omega_0 t\right)\right]$$

x_h: $\ddot{x} + 2\gamma\dot{x} + \omega_0^2 x = 0$

$$\ddot{x} + \frac{2\omega_0}{3}\dot{x} + \omega_0^2 x = 0 \qquad x = e^{pt}$$

$$p^2 + \frac{2}{3}\omega_0 p + \omega_0^2 = 0$$

$$p = \frac{-\frac{2}{3}\omega_0 \pm \left(\frac{4}{9}\omega_0^2 - 4\omega_0^2\right)^{1/2}}{2}$$

$$p = -\frac{\omega_0}{3} \pm \left(\frac{1}{9}\omega_0^2 - \omega_0^2\right)^{1/2}$$

$$= -\frac{\omega_0}{3} \pm \left(-\frac{8}{9}\omega_0^2\right)^{1/2}$$

$$= -\frac{\omega_0}{3} \pm i\,\frac{2\sqrt{2}}{3}\omega_0$$

$$x_h = e^{-\frac{\omega_0}{3}t}\left(C_1\cos\left(\frac{2\sqrt{2}}{3}\omega_0 t\right) + C_2\sin\left(\frac{2\sqrt{2}}{3}\omega_0 t\right)\right)$$

$$x(t) = e^{-\frac{\omega_0}{3}t}\left(C_1\cos\left(\frac{2\sqrt{2}}{3}\omega_0 t\right) + C_2\sin\left(\frac{2\sqrt{2}}{3}\omega_0 t\right)\right) - \frac{1}{m\omega_0^2}\left[\frac{3A}{2}\cos\omega_0 t + \frac{B}{34}\left(\cos 3\omega_0 t - 4\sin(3\omega_0 t)\right)\right]$$

$$0 = x(0) = C_1 - \frac{1}{m\omega_0^2}\left[\frac{3A}{2} + \frac{B}{34}\right]$$

$$C_1 = \frac{1}{m\omega_0^2}\left[\frac{3A}{2} + \frac{B}{34}\right]$$

$$\dot{x}(t) = -\frac{\omega_0}{3} e^{-\frac{\omega_0}{3}t}\left(C_1 \cos\frac{2\sqrt{2}}{3}\omega_0 t + C_2 \sin\frac{2\sqrt{2}}{3}\omega_0 t\right) + e^{-\frac{\omega_0}{3}t}\left(-\frac{2\sqrt{2}\omega_0}{3}\sin\frac{2\sqrt{2}}{3}\omega_0 t + \frac{2\sqrt{2}\omega_0}{3}\cos\frac{2\sqrt{2}}{3}\omega_0 t\right) -$$

$$\frac{1}{m\omega_0^2}\left[-\frac{3A\omega_0}{2}\sin\omega_0 t - \frac{3B\omega_0}{34}\sin 3\omega_0 t + \frac{12B\omega_0}{34}\cos 3\omega_0 t\right]$$

$$0 = \dot{x}(0) = -\frac{\omega_0}{3}C_1 + \frac{2\sqrt{2}}{3}\omega_0 C_2 - \frac{1}{m\omega_0^2}\frac{12B\omega_0}{34}$$

$$\frac{C_1}{3} + \frac{1}{m\omega_0^2}\frac{12B}{34} = \frac{2\sqrt{2}}{3}C_2$$

$$\frac{1}{3}\left(\frac{1}{m\omega_0^2}\left[\frac{3A}{2} + \frac{B}{34}\right]\right) + \frac{1}{m\omega_0^2}\frac{12B}{34} = \frac{2\sqrt{2}}{3}C_2$$

$$\frac{1}{m\omega_0^2}\left[\frac{A}{2} + \frac{B}{102} + \frac{12B}{34}\right] = \frac{2\sqrt{2}}{3}C_2$$

$$\frac{1}{m\omega_0^2}\frac{3}{2\sqrt{2}}\left[\frac{A}{2} + \frac{B}{102} + \frac{12B}{34}\right] = C_2$$

$$\frac{1}{m\omega_0^2}\frac{3\sqrt{2}}{4}\left[\frac{A}{2} + \frac{B}{102} + \frac{12B}{34}\right] = C_2$$

$$\frac{1}{m\omega_0^2}\left[\frac{3\sqrt{2}}{8}A + \frac{\sqrt{2}}{136}B + \frac{36\sqrt{2}}{136}B\right] = C_2$$

$$\frac{1}{m\omega_0^2}\left[\frac{3\sqrt{2}}{8}A + \frac{37\sqrt{2}}{136}B\right] = C_2$$

$$\boxed{x(t) = \frac{1}{m\omega_0^2}\left[e^{-\frac{\omega_0 t}{3}}\left[\left(\frac{3}{2}A + \frac{B}{34}\right)\cos\frac{2\sqrt{2}}{3}\omega_0 t + \left(\frac{3\sqrt{2}}{8}A + \frac{37\sqrt{2}}{136}B\right)\sin\frac{2\sqrt{2}}{3}\omega_0 t\right] - \frac{3A}{2}\cos\omega_0 t - \frac{B}{34}\cos 3\omega_0 t - \frac{2B}{17}\sin 3\omega_0 t\right]}$$

b) $$x(t) = \frac{1}{m\omega_0^2}\left[-\frac{3A}{2}\cos\omega_0 t - \frac{B}{34}\left(\cos 3\omega_0 t + 4\sin 3\omega_0 t\right)\right], \quad t >> 0$$

$$\frac{3A}{2}\cos\omega_0 t = \frac{B}{34}\left(\cos 3\omega_0 t + 4\sin 3\omega_0 t\right)$$

$$\boxed{\frac{B}{A} = \frac{51\cos\omega_0 t}{\cos 3\omega_0 t + 4\sin 3\omega_0 t}}$$

2-59

$$m\ddot{x} + b\dot{x} + kx = F_0 - F_0 e^{-at}, \quad k = ma^2,\ b = 2ma$$

$$\ddot{x} + \frac{b}{m}\dot{x} + \frac{k}{m}x = \frac{F_0}{m} - \frac{F_0}{m} e^{-at}$$

$$\ddot{x} + 2a\dot{x} + a^2 x = \frac{F_0}{m} - \frac{F_0}{m} e^{-at}$$

$$x_h: \quad \ddot{x} + 2a\dot{x} + a^2 x = 0 \qquad x = e^{pt}$$

$$p^2 + 2ap + a^2 = 0$$

$$p = \frac{-2a \pm (4a^2 - 4a^2)}{2}$$

$$= -a$$

$$x_h = C_1 e^{-at} + C_2 t e^{-at}$$

$$x_p = K_1$$

$$a^2 K_1 = \frac{F_0}{m}$$

$$K_1 = \frac{F_0}{ma^2}$$

$$x_1 = (C_1 + C_2 t) e^{-at} + \frac{F_0}{ma^2}$$

$$0 = x(0) = C_1 + \frac{F_0}{ma^2}$$

$$C_1 = -\frac{F_0}{ma^2}$$

$$\dot{x}_1 = -a(C_1 + C_2 t) e^{-at} + C_2 e^{-at}$$

$$0 = v(0) = -aC_1 + C_2$$

$$C_2 = aC_1$$

$$= -\frac{F_0}{ma}$$

$$x_1 = -\frac{F_0}{ma}\left(\frac{1}{a} + t\right) e^{-at} + \frac{F_0}{ma^2}$$

$$x_h = K_1 e^{-at} + K_2 t e^{-at}$$

$$x_{p_2} = K_3 t^2 e^{-at} \quad \{\text{guess method}\}$$

$$\dot{x}_{p_2} = -aK_3 t^2 e^{-at} + 2K_3 t e^{-at}$$

$$\ddot{x}_{p_2} = a^2 K_3 t^2 e^{-at} - 2aK_3 t e^{-at} - 2aK_3 t e^{-at} + 2K_3 e^{-at}$$

$$= a^2 K_3 t^2 e^{-at} - 4aK_3 t e^{-at} + 2K_3 e^{-at}$$

$$\ddot{x} + 2a\dot{x} + a^2 x = -\frac{F_0}{m} e^{-at}$$

$$a^2 K_3 t^2 - 4aK_3 t + 2K_3 + 2a\left(-aK_3 t^2 e^{-at} + 2K_3 t\right) + a^2 K_3 t^2 = -\frac{F_0}{m}$$

$$t^2\left(a^2 K_3 - 2a^2 K_3 + a^2 K_3\right) + t\left(-4aK_3 + 4aK_3\right) + 2K_3 = -\frac{F_0}{m}$$

$$K_3 = \frac{-F_0}{2m}$$

$$x_2 = K_1 e^{-at} + K_2 t e^{-at} - \frac{F_0}{2m} t^2 e^{-at}$$

$$0 = x_2(0) = K_1$$

$$\dot{x}_2 = -aK_2 t e^{-at} + K_2 e^{-at} + \frac{F_0 a}{2m} t^2 e^{-at} - \frac{F_0 t}{2m} e^{-at}$$

$$0 = v_2(0) = K_2$$

$$x_2 = -\frac{F_0}{2m} t^2 e^{-at}$$

$$x = x_1 + x_2$$

$$= -\frac{F_0}{ma}\left(\frac{1}{a} + t\right) e^{-at} + \frac{F_0}{ma^2} - \frac{F_0}{2m} t^2 e^{-at}$$

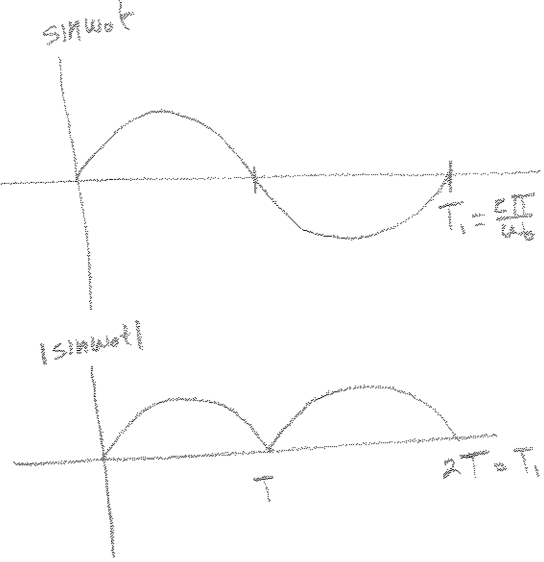
$$= \frac{-F_0}{ma^2} e^{-at} - \frac{F_0}{ma} t e^{-at} + \frac{F_0}{ma^2} - \frac{F_0}{2m} t^2 e^{-at}$$

$$\boxed{x = \frac{F_0}{ma^2}\left[1 - \left(1 + at + \frac{a^2}{2} t^2\right) e^{-at}\right]}$$

2-61 $\ddot{x} + \omega_0^2 x = F_0 |\sin\omega_0 t|$

$$F(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{a} \quad \{\text{period} = 2a\}$$

$b_n = 0$ {even function}



$$a_n = \frac{1}{a}\int_{-a}^{a} F(t)\cos\frac{n\pi t}{a}\,dt$$

, period of rectified sine is one half of sine

$$T = \frac{T_1}{2} = \frac{\pi}{\omega_0} = 2a$$

$$= \frac{2F_0}{T}\int_0^T \sin\frac{\pi t}{T}\cos\frac{2\pi n t}{T}\,dt$$

$$= \frac{2F_0}{T}\int_0^T \left(\frac{e^{i\pi t/T} - e^{-i\pi t/T}}{2i}\right)\left(\frac{e^{i2\pi n t/T} + e^{-i2\pi n t/T}}{2}\right)dt$$

$$= \frac{F_0}{2Ti}\int_0^T \left[e^{i\pi t/T(1+2n)} - e^{i\pi t/T(2n-1)} + e^{i\pi t/T(1-2n)} - e^{-i\pi t/T(1+2n)}\right]dt$$

$$= \frac{F_0}{2Ti}\left[\frac{e^{i\pi t/T(1+2n)}}{\frac{i\pi(1+2n)}{T}} - \frac{e^{i\pi t/T(2n-1)}}{\frac{i\pi(2n-1)}{T}} + \frac{e^{i\pi t/T(1-2n)}}{\frac{i\pi(1-2n)}{T}} + \frac{e^{-i\pi t/T(1+2n)}}{\frac{i\pi(1+2n)}{T}}\right]_0^T$$

$$= \frac{-F_0}{2\pi}\left[\frac{e^{i\pi(1+2n)}}{1+2n} - \frac{e^{i\pi(2n-1)}}{2n-1} + \frac{e^{i\pi(1-2n)}}{1-2n} + \frac{e^{-i\pi(1+2n)}}{1+2n} - \left(\frac{1}{1+2n} - \frac{1}{2n-1} + \frac{1}{1-2n} + \frac{1}{1+2n}\right)\right]$$

, $e^{i\pi} = -1 = e^{-i\pi}$, $e^{i2\pi n} = 1 = e^{-i2\pi n}$

$$= -\frac{F_0}{2\pi}\left[\frac{-1}{1+2n} + \frac{1}{2n-1} - \frac{1}{1-2n} - \frac{1}{1+2n} - \frac{1}{1+2n} + \frac{1}{2n-1} - \frac{1}{1-2n} - \frac{1}{1+2n}\right]$$

$$= -\frac{F_0}{2\pi}\left[\frac{-2}{1+2n} + \frac{2}{2n-1} - \frac{2}{1-2n} - \frac{2}{1+2n}\right]$$

$$= -\frac{F_0}{2\pi}\left[\frac{-4}{1+2n} - \frac{4}{1-2n}\right]$$

$$= \frac{2F_0}{\pi}\left[\frac{1}{1+2n} + \frac{1}{1-2n}\right]$$

$$= -\frac{2F_0}{\pi}\left[\frac{1-2n+1+2n}{4n^2-1}\right]$$

$$= \frac{-2F_0}{\pi}\left[\frac{2}{4n^2-1}\right]$$

$$a_n = -\frac{4F_0}{\pi}\frac{1}{4n^2-1}$$

$$a_0 = \frac{1}{2a}\int_{-a}^{a} f(t)\,dt \qquad a = \frac{T}{2}$$

$$= \frac{F_0}{T}\int_0^T \sin\frac{\pi t}{T}\,dt$$

$$= \frac{F_0}{T}\left[\frac{-T}{\pi}\cos\frac{\pi t}{T}\right]\Big|_0^T$$

$$= \frac{-F_0}{\pi}\left[\cos\pi - \cos 0\right]$$

$$= \frac{-F_0}{\pi}\left[-1-1\right]$$

$$= \frac{2F_0}{\pi}$$

$$F(t) = \frac{2F_0}{\pi} - \frac{4F_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos\frac{2\pi n t}{T}$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n\cos\frac{2\pi n t}{T} + B_n\sin\frac{2\pi n t}{T}$$

$$\dot{x}(t) = \sum_{n=1}^{\infty} -\frac{2\pi n}{T}A_n\sin\frac{2\pi n t}{T} + \frac{2\pi n}{T}B_n\cos\frac{2\pi n t}{T}$$

$$\ddot{x}(t) = \sum_{n=1}^{\infty} -\left(\frac{2\pi n}{T}\right)^2 A_n\cos\frac{2\pi n t}{T} - \left(\frac{2\pi n}{T}\right)^2 B_n\sin\frac{2\pi n t}{T}$$

$$\ddot{x} + \omega_0^2 x = F(t)$$

$$\sum_{n=1}^{\infty}\left(-\left(\frac{2\pi n}{T}\right)^2 A_n\cos\frac{2\pi n t}{T} - \left(\frac{2\pi n}{T}\right)^2 B_n\sin\frac{2\pi n t}{T}\right) + \omega_0^2\left(\sum_{n=1}^{\infty} A_n\cos\frac{2\pi n t}{T} + B_n\sin\frac{2\pi n t}{T}\right) + A_0 =$$

$$\frac{2F_0}{\pi} - \frac{4F_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{4n^2-1}\cos\frac{2\pi n t}{T}$$

Equating terms:

$$\omega_0^2 A_0 = \frac{2F_0}{\pi} \rightarrow A_0 = \frac{2F_0}{\pi\omega_0^2}$$

$$-\left(\frac{2\pi n}{T}\right)^2 A_n + \omega_0^2 A_n = -\frac{4F_0}{\pi}\frac{1}{4n^2-1} \qquad \omega_0 = \frac{\pi}{T}$$

$$-(2\omega_0 n)^2 A_n + \omega_0^2 A_n = -\frac{4F_0}{\pi}\frac{1}{4n^2-1}$$

$$-4\omega_0^2 n^2 A_n + \omega_0^2 A_n = -\frac{4F_0}{\pi}\frac{1}{4n^2-1}$$

$$A_n \omega_0^2 (4n^2-1) = \frac{4F_0}{\pi}\frac{1}{4n^2-1}$$

$$A_n = \frac{4F_0}{\pi\omega_0^2}\frac{1}{(4n^2-1)^2}$$

$$x(t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{2\pi n t}{T}$$

$$\boxed{x(t) = \frac{2F_0}{\pi\omega_0^2} + \frac{4F_0}{\pi\omega_0^2}\sum_{n=1}^{\infty}\frac{\cos 2n\omega_0 t}{(4n^2-1)^2}}$$

3-1

a) Prove: $c(\vec{A}+\vec{B}) = c\vec{A} + c\vec{B}$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$
$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

$$\vec{A}+\vec{B} = (A_x+B_x)\hat{x} + (A_y+B_y)\hat{y} + (A_z+B_z)\hat{z}$$

$$c(\vec{A}+\vec{B}) = c(A_x+B_x)\hat{x} + c(A_y+B_y)\hat{y} + c(A_z+B_y)\hat{z}$$
$$= cA_x\hat{x} + cA_y\hat{y} + cA_z\hat{z} + cB_x\hat{x} + cB_y\hat{y} + cB_z\hat{z}$$
$$= c(A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) + c(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

$$c(\vec{A}+\vec{B}) = c\vec{A} + c\vec{B}$$

b) $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot((B_x+C_x)\hat{x} + (B_y+C_y)\hat{y} + (B_z+C_z)\hat{z})$$
$$= (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\cdot((B_x+C_x)\hat{x} + (B_y+C_y)\hat{y} + (B_z+C_z)\hat{z})$$
$$= A_x(B_x+C_x) + A_y(B_y+C_y) + A_z(B_z+C_z)$$
$$= A_xB_x + A_yB_y + A_yB_z + A_xC_x + A_yC_y + A_zC_z$$

$$\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B} + \vec{A}\cdot\vec{C}$$

c) $\vec{A}\times(\vec{B}+\vec{C}) = (\vec{A}\times\vec{B}) + (\vec{A}\times\vec{C})$

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times((B_x+C_x)\hat{x} + (B_y+C_y)\hat{y} + (B_z+C_z)\hat{z})$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ B_x+C_x & B_y+C_y & B_z+C_z \end{vmatrix}$$

$$= \hat{x}(A_y(B_z+C_z) - A_z(B_y+C_y)) + \hat{y}(A_z(B_x+C_x) - A_x(B_z+C_z)) + \hat{z}(A_x(B_y+C_y) - A_y(B_x+C_x))$$

$$= \hat{x}(A_yB_z - A_zB_y + A_yC_z - A_zC_y) +$$
$$\hat{y}(A_zB_x - A_xB_z + A_zC_x - A_xC_z) +$$
$$\hat{z}(A_xB_y - A_yB_x + A_xC_y - A_yC_x)$$

$$= \hat{x}(A_yB_z - A_zB_y) + \hat{y}(A_zB_x - A_xB_z) + \hat{z}(A_xB_y - A_yB_x) +$$
$$\hat{x}(A_yC_z - A_zC_y) + \hat{y}(A_zC_x - A_xC_z) + \hat{z}(A_xC_y - A_yC_x)$$

$$\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}$$

d) $\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{A}\times\left((B_yC_z - B_zC_y)\hat{x} + (B_zC_x - B_xC_z)\hat{y} + (B_xC_y - B_yC_x)\hat{z}\right)$$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ A_x & A_y & A_z \\ (B_yC_z - B_zC_y) & B_zC_x - B_xC_z & B_xC_y - B_yC_x \end{vmatrix}$$

$$= \hat{x}\left[A_y(B_xC_y - B_yC_x) - A_z(B_zC_x - B_xC_z)\right] +$$
$$\hat{y}\left[A_z(B_yC_z - B_zC_y) - A_x(B_xC_y - B_yC_x)\right] +$$
$$\hat{z}\left[A_x(B_zC_x - B_xC_z) - A_y(B_yC_z - B_zC_y)\right]$$

$$= \hat{x}\left[B_x(A_yC_y) - C_x(A_yB_y) - C_x(A_zB_z) + B_x(A_zC_z)\right]$$
$$\hat{y}\left[B_y(A_zC_z) - C_y(A_zB_z) - C_y(A_xB_x) + B_y(A_xC_x)\right]$$
$$\hat{z}\left[B_z(A_xC_x) - C_z(A_xB_x) - C_z(A_yB_y) + B_z(A_yC_y)\right]$$

$$= \hat{x}\left[B_x[A_yC_y + A_zC_z] - C_x[A_yB_y + A_zB_z]\right] +$$
$$\hat{y}\left[B_y[A_zC_z + A_xC_x] - C_y[A_zB_z + A_xB_x]\right] +$$
$$\hat{z}\left[B_z[A_xC_x + A_yC_y] - C_z[A_xB_x + A_yB_y]\right]$$

$$= \hat{x}\left[B_x[A_xC_x + A_yC_y + A_zC_z - A_xC_x] - C_x[A_xB_x + A_yB_y + A_zB_z - A_xB_x]\right] +$$
$$\hat{y}\left[B_y[A_xC_x + A_yC_y + A_zC_z - A_yC_y] - C_y[A_xB_x + A_yB_y + A_zB_z - A_yB_y]\right] +$$
$$\hat{z}\left[B_z[A_xC_x + A_yC_y + A_z z - A_zC_z] - C_z[A_xB_x + A_yB_y + A_zB_z - A_zB_z]\right]$$

$$= \hat{x}\left[B_x\left[(\vec{A}\cdot\vec{C}) - A_xC_x\right] - C_x\left[(\vec{A}\cdot\vec{B}) + A_xB_x\right]\right] +$$

$$\hat{y}\left[B_y\left[(\vec{A}\cdot\vec{C}) - A_yC_y\right] - C_y\left[(\vec{A}\cdot\vec{B}) + A_yB_z\right]\right] +$$

$$\hat{z}\left[B_z\left[(\vec{A}\cdot\vec{C}) - A_zC_z\right] - C_z\left[(\vec{A}\cdot\vec{B}) + A_zB_z\right]\right]$$

$$= \hat{x}\left[B_x(\vec{A}\cdot\vec{C}) - A_xB_xC_x - C_x(\vec{A}\cdot\vec{B}) + A_xB_xC_x\right] +$$

$$\hat{y}\left[B_y(\vec{A}\cdot\vec{C}) - A_yB_yC_y - C_y(\vec{A}\cdot\vec{B}) + A_yB_yC_y\right] +$$

$$\hat{z}\left[B_z(\vec{A}\cdot\vec{C}) - A_zB_zC_z - C_z(\vec{A}\cdot\vec{B}) + A_zB_zC_z\right]$$

$$\vec{A}\times(\vec{B}\times\vec{C}) = \vec{B}(\vec{A}\cdot\vec{C}) - \vec{C}(\vec{A}\cdot\vec{B})$$

3-3 Prove $\vec{A}\times\vec{B} = (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x)$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$\vec{B} = B_x\hat{x} + B_y\hat{y} + B_z\hat{z}$$

$$\vec{A}\times\vec{B} = \vec{A}\times(B_x\hat{x} + B_y\hat{y} + B_z\hat{z})$$

$$= \vec{A}\times B_x\hat{x} + \vec{A}\times B_y\hat{y} + \vec{A}\times B_z\hat{z}$$

$$= (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\times B_x\hat{x} + (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\times B_y\hat{y} + (A_x\hat{x} + A_y\hat{y} + A_z\hat{z})\times B_z\hat{z}$$

$$= A_x\hat{x}\times B_x\hat{x} + A_y\hat{y}\times B_x\hat{x} + A_z\hat{z}\times B_x\hat{x} + \qquad \{\vec{A}\times(\vec{B}+\vec{C}) = \vec{A}\times\vec{B} + \vec{A}\times\vec{C}\}$$

$$A_x\hat{x}\times B_y\hat{y} + A_y\hat{y}\times B_y\hat{y} + A_z\hat{z}\times B_y\hat{y} +$$

$$A_x\hat{x}\times B_z\hat{z} + A_y\hat{y}\times B_z\hat{z} + A_z\hat{z}\times B_z\hat{z}$$

$$= \vec{0} + A_yB_x(\hat{y}\times\hat{x}) + A_zB_x(\hat{z}\times\hat{x}) + \qquad \{|\vec{A}\times\vec{B}| = AB, (c\vec{A})\times\vec{B} = \vec{A}\times(c\vec{B}) =$$

$$A_xB_y(\hat{x}\times\hat{y}) + \vec{0} + A_zB_y(\hat{z}\times\hat{y}) + \qquad c(\vec{A}\times\vec{B})\}$$

$$A_xB_z(\hat{x}\times\hat{z}) + A_yB_z(\hat{y}\times\hat{z}) + \vec{0}$$

$$= -A_yB_x\hat{z} + A_zB_x\hat{y} + \qquad \{\vec{A}\times\vec{B} = -\vec{B}\times\vec{A},\ \hat{x}\times\hat{x} = \hat{y}\times\hat{y} = \hat{z}\times\hat{z} = \vec{0},$$

$$\hat{x}\times\hat{y} = \hat{z},\ \hat{y}\times\hat{z} = \hat{x},\ \hat{z}\times\hat{x} = \hat{y}\}$$

$$A_xB_y\hat{z} - A_zB_y\hat{x} +$$

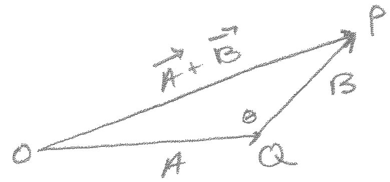
$$-A_xB_z\hat{y} + A_yB_z\hat{x}$$

$$= (A_yB_z - A_zB_y)\hat{x} + (A_zB_x - A_xB_z)\hat{y} + (A_xB_y - A_yB_x)\hat{z}$$

$$\vec{A}\times\vec{B} = (A_yB_z - A_zB_y, A_zB_x - A_xB_z, A_xB_y - A_yB_x)$$

3-5 a) Prove $|\vec{A}+\vec{B}| \le |\vec{A}| + |\vec{B}|$

Geometric Proof:



The shortest distance between O and P is a straight line. The length $|\vec{A}+\vec{B}|$ is the shortest distance. Any other path, O-Q-P has a length of $|\vec{A}| + |\vec{B}|$. If Q is on $\overline{OP}$ then $|\vec{A}| + |\vec{B}| = |\vec{A}+\vec{B}|$ otherwise $|\vec{A}| + |\vec{B}| > |\vec{A}+\vec{B}|$.

Algebraic Proof:

$$|\vec{A}| = \sqrt{A_x^2 + A_y^2}$$

$$|\vec{B}| = \sqrt{B_x^2 + B_y^2}$$

$$|\vec{A}+\vec{B}| = \sqrt{(A_x+B_x)^2 + (A_y+B_y)^2}$$

$$|\vec{A}| + |\vec{B}| = \sqrt{A_x^2 + A_y^2} + \sqrt{B_x^2 + B_y^2}$$

$$(|\vec{A}+\vec{B}|)^2 = (A_x+B_x)^2 + (A_y+B_y)^2$$

$$(|\vec{A}|+|\vec{B}|)^2 = A_x^2 + A_y^2 + B_x^2 + B_y^2 + 2\sqrt{(A_x^2+A_y^2)(B_x^2+B_y^2)}$$

$$(|\vec{A}+\vec{B}|)^2 = A_x^2 + 2A_xB_x + B_x^2 + A_y^2 + 2A_yB_y + B_y^2$$
$$= A_x^2 + A_y^2 + B_x^2 + B_y^2 + 2A_xB_x + 2A_yB_y$$

$$(|\vec{A}|+|\vec{B}|)^2 - (|\vec{A}+\vec{B}|)^2 = 2\sqrt{(A_x^2+A_y^2)(B_x^2+B_y^2)} - 2(A_xB_x + A_yB_y)$$
$$= 2\sqrt{(A_xB_x)^2 + (A_xB_y)^2 + (A_yB_x)^2 + (A_yB_y)^2} - 2(A_xB_x + A_yB_y)$$

Compare the differnce terms:

$$2\sqrt{(A_xB_x)^2 + (A_xB_y)^2 + (A_yB_x)^2 + (A_yB_y)^2} \overset{?}{=} 2(A_xB_x + A_yB_y)$$

$$(A_xB_x)^2 + (A_xB_y)^2 + (A_yB_x)^2 + (A_yB_y)^2 \overset{?}{=} (A_xB_x + A_yB_y)^2$$
$$\overset{?}{=} (A_xB_x)^2 + 2(A_xA_yB_xB_y) + (A_yB_y)^2$$

$$(A_xB_y)^2 + (A_yB_x)^2 \overset{?}{=} 2A_xA_yB_xB_y$$

$$\frac{(A_xB_y)^2 + (A_yB_x)^2}{A_xA_yB_xB_y} \overset{?}{=} 2$$

$$\frac{A_x B_y}{A_y B_x} + \frac{A_y B_x}{A_x B_y} \stackrel{?}{\geq} 2$$

Let $B_x = \alpha A_x$ and $B_y = \gamma A_y$

$$\frac{A_x \gamma A_y}{A_y \alpha A_x} + \frac{A_y \alpha A_x}{A_x \gamma A_y} \stackrel{?}{\geq} 2$$

$$\frac{\gamma}{\alpha} + \frac{\alpha}{\gamma} \stackrel{?}{\geq} 2$$

$$\gamma^2 + \alpha^2 \stackrel{?}{\geq} 2\alpha\gamma$$

$$\gamma^2 - 2\alpha\gamma + \alpha^2 \stackrel{?}{\geq} 0$$

$$(\gamma-\alpha)^2 \stackrel{?}{\geq} 0$$

$$(\gamma-\alpha)^2 \geq 0$$

So,

$$\gamma^2 - 2\alpha\gamma + \alpha^2 \geq 0$$

$$\frac{\gamma^2 + \alpha^2}{\alpha\gamma} \geq 2$$

$$\frac{\gamma}{\alpha} + \frac{\alpha}{\gamma} \geq 2$$

Therefore,

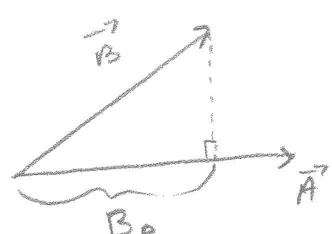
$$(|\vec{A}| + |\vec{B}|)^2 - (|\vec{A} + \vec{B}|)^2 \geq 0$$

and

$$|\vec{A}| + |\vec{B}| \geq |\vec{A} + \vec{B}|.$$

b) $|\vec{A} \cdot \vec{B}| \leq |\vec{A}||\vec{B}|$

Geometric Proof:



$|\vec{A} \cdot \vec{B}| = |\vec{A}||B_p|$ where B_p is the projection of $\vec{B}$ on $\vec{A}$. $B_p \leq |\vec{B}|$ since B_p is a leg of a right triangle with hypotenuse $|\vec{B}|$. So, $|\vec{A}||\vec{B_p}| \leq |\vec{A}||\vec{B}|$.
Using substitution $|\vec{A} \cdot \vec{B}| \leq |\vec{A}||\vec{B}|$

Algebraic Proof:

$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta$

$|\vec{A} \cdot \vec{B}| = |\vec{A}||\vec{B}||\cos\theta|$

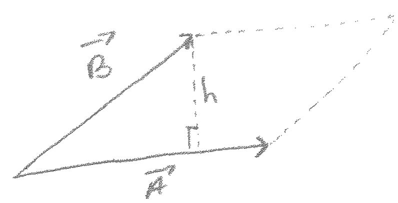
$|\cos\theta| \leq 1 \rightarrow |\vec{A}||\vec{B}||\cos\theta| \leq |\vec{A}||\vec{B}|$

Therefore, by substitution

$|\vec{A} \cdot \vec{B}| \leq |\vec{A}||\vec{B}|$

c) $|\vec{A} \times \vec{B}| \leq |\vec{A}||\vec{B}|$

Geometric Proof:



$|\vec{A} \times \vec{B}| = h|\vec{A}|$. Since h is a leg of a right triangle with hypotenuse $|\vec{B}|$ then $h \leq |\vec{B}|$. So, $h|\vec{A}| \leq |\vec{A}||\vec{B}|$. Using substitution

$$|\vec{A} \times \vec{B}| \leq |\vec{A}||\vec{B}|$$

Algebraic Proof:

$\vec{A} \times \vec{B} = |\vec{A}||\vec{B}|\sin\theta$

$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}||\sin\theta|$

$|\sin\theta| \leq 1 \rightarrow |\vec{A}||\vec{B}||\sin\theta| \leq |\vec{A}||\vec{B}|$

Therefore, by substitution

$|\vec{A} \times \vec{B}| \leq |\vec{A}||\vec{B}|$

3-7 i) Prove

$$\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}.$$

Defn: $\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$

$$\frac{d}{dt}(\vec{A}+\vec{B}) = \lim_{\Delta t \to 0} \frac{(\vec{A}(t+\Delta t) + \vec{B}(t+\Delta t)) - [\vec{A}(t) + \vec{B}(t)]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t) + \vec{B}(t+\Delta t) - \vec{B}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t} + \lim_{\Delta t \to 0} \frac{\vec{B}(t+\Delta t) - \vec{B}(t)}{\Delta t}$$

$$\frac{d}{dt}(\vec{A}+\vec{B}) = \frac{d\vec{A}}{dt} + \frac{d\vec{B}}{dt}$$

ii) Prove

$$\frac{d}{dt}(f\vec{A}) = \frac{df}{dt}\vec{A} + f\frac{d\vec{A}}{dt}$$

Defn: $\frac{d\vec{A}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t}$

$$\frac{d}{dt}(f\vec{A}) = \lim_{\Delta t \to 0} \frac{f(t+\Delta t)\vec{A}(t+\Delta t) - f(t)\vec{A}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t)\vec{A}(t+\Delta t) - f(t+\Delta t)\vec{A}(t) + f(t+\Delta t)\vec{A}(t) - f(t)\vec{A}(t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t)[\vec{A}(t+\Delta t) - \vec{A}(t)] + \vec{A}(t)[f(t+\Delta t) - f(t)]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{f(t+\Delta t)[\vec{A}(t+\Delta t) - \vec{A}(t)]}{\Delta t} + \lim_{\Delta t \to 0} \frac{\vec{A}(t)[f(t+\Delta t) - f(t)]}{\Delta t}$$

$$= \lim_{\Delta t \to 0} f(t+\Delta t) \lim_{\Delta t \to 0} \frac{\vec{A}(t+\Delta t) - \vec{A}(t)}{\Delta t} + \lim_{\Delta t \to 0} \vec{A}(t) \lim_{\Delta t \to 0} \frac{f(t+\Delta t) - f(t)}{\Delta t}$$

$$\frac{d}{dt}(f\vec{A}) = f\frac{d\vec{A}}{dt} + \vec{A}\frac{df}{dt}$$

3-9

Defn: $\displaystyle\int_{t_1}^{t_2} \vec{A}(t)\,dt = \lim_{n\to\infty} \sum_{i=1}^{n} \vec{A}(t_i)\,\Delta t_i \,, \quad \Delta t_i = \frac{t_2 - t_1}{n}$

$$\int_{t_1}^{t_2} \vec{A}(t)\,dt = \hat{x}\int_{t_1}^{t_2} A_x\,dt + \hat{y}\int_{t_1}^{t_2} A_y\,dt + \hat{z}\int_{t_1}^{t_2} A_z\,dt$$

Analogous equations (3.54)-(3.57):

3.54: $\displaystyle\int_{t_1}^{t_2} (\vec{A}+\vec{B})\,dt = \lim_{n\to\infty}\sum_{i=1}^{n}(\vec{A}(t_i)+\vec{B}(t_i))\,\Delta t_i$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\vec{A}(t_i)\,\Delta t_i + \lim_{n\to\infty}\sum_{i=1}^{n}\vec{B}(t_i)\,\Delta t_i$$

$$= \int_{t_1}^{t_2}\vec{A}(t)\,dt + \int_{t_1}^{t_2}\vec{B}(t)\,dt$$

3.55: $\displaystyle\int_{t_1}^{t_2} f\vec{A}\,dt = \lim_{n\to\infty}\sum_{i=1}^{n} f(t_i)\vec{A}(t_i)\,\Delta t_i$

Let $\vec{G}(t) = f\vec{A}(t)$

$$\int_{t_1}^{t_2} f\vec{A}\,dt = \lim_{n\to\infty}\sum_{i=1}^{n}\vec{G}(t_i)\,\Delta t_i$$

$$= \int_{t_1}^{t_2}\vec{G}(t)\,dt = \hat{x}\int_{t_1}^{t_2} fA_x\,dt + \hat{y}\int_{t_1}^{t_2} fA_y\,dt + \hat{z}\int_{t_1}^{t_2} fA_z\,dt$$

3.56: $\displaystyle\int_{t_1}^{t_2}(\vec{A}\cdot\vec{B})\,dt = \int_{t_1}^{t_2} A_xB_x\,dt + \int_{t_1}^{t_2} A_yB_y\,dt + \int_{t_1}^{t_2} A_zB_z\,dt$

3.57: $\displaystyle\int_{t_1}^{t_2}(\vec{A}\times\vec{B})\,dt = \int_{t_1}^{t_2}\vec{C}\,dt = \hat{x}\int_{t_1}^{t_2}(A_yB_z - B_zA_y)\,dt + \hat{y}\int_{t_1}^{t_2}(A_zB_x - A_xB_z)\,dt + \hat{z}\int_{t_1}^{t_2}(A_xB_y - A_yB_x)\,dt$

Prove:

$$\frac{d}{dt}\int_0^t \vec{A}(t)\,dt = \vec{A}(t)$$

$\vec{A}(t)$ is continuos. Then

$$\vec{a}(t) = \int_0^t \vec{A}(t)\,dt$$

Must show

$$\frac{d\vec{a}}{dt} = \vec{A}(t).$$

$$\vec{a}(t+\Delta t) = \int_0^{t+\Delta t} \vec{A}(t)\,dt$$

$$\vec{a}(t+\Delta t) - \vec{a}(t) = \int_0^{t+\Delta t} \vec{A}(t)\,dt - \int_0^{t} A(t)\,dt$$

$$= \int_t^{t+\Delta t} \vec{A}(t)\,dt$$

The Mean Value Theorem states

$$\int_t^{t+\Delta t} \vec{A}(t)\,dt = \vec{A}(c)\Delta t$$ (a value $A(c)$ is between $A(t+\Delta t)$ and $A(t)$)

$$\vec{a}(t+\Delta t) - \vec{a}(t) = \vec{A}(c)\Delta t$$

$$\frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t} = \vec{A}(c)$$

$$\lim_{\Delta t \to 0} \frac{\vec{a}(t+\Delta t) - \vec{a}(t)}{\Delta t} = \lim_{\Delta t \to 0} \vec{A}(c)$$

$$\frac{d\vec{a}}{dt} = \lim_{\Delta t \to 0} \vec{A}(c)$$

Since c is between t and $t+\Delta t$, and

$$\lim_{\Delta t \to 0} t = t \,, \quad \lim_{\Delta t \to 0} t+\Delta t = t$$

then

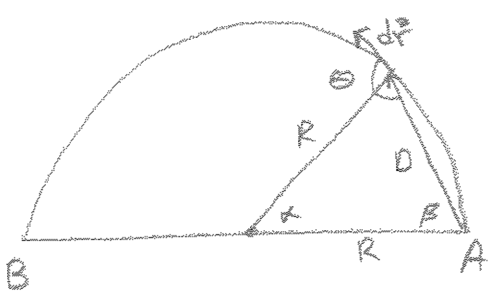
$$\lim_{\Delta t \to 0} c = t$$ {squeeze theorem}

Finally,

$$\frac{d\vec{a}}{dt} = \lim_{\Delta t \to 0} \vec{A}(c)$$

$$\frac{d}{dt}\int_0^t \vec{A}(t)\,dt = A(t)$$

3-11



$$\theta = \beta + \frac{\pi}{2}$$

$$2\beta + \alpha = \pi$$

$$\beta = \frac{\pi}{2} - \frac{\alpha}{2}$$

$$\theta = \pi - \frac{\alpha}{2}$$

$$d\vec{s} = \hat{r}\,dr$$

$$ds = R\,d\alpha$$

$$F = -KD$$

$$D^2 = R^2 + R^2 - 2R^2\cos\alpha$$

$$= 2R^2(1-\cos\alpha)$$

$$D = \sqrt{2}R(1-\cos\alpha)^{1/2}$$

$$= 2R\sin\frac{\alpha}{2}$$

$$\sin^2\gamma = \frac{1}{2}(1-\cos 2\gamma)$$

$$\sin\gamma = \frac{1}{\sqrt{2}}(1-\cos 2\gamma)^{1/2}$$

$$\sqrt{2}\sin\frac{\gamma}{2} = (1-\cos\gamma)^{1/2}$$

At $D = 2R$ $F = F_0$

$$F_0 = -K2R$$

$$-K = \frac{F_0}{2R}$$

$$F = \frac{-F_0}{2R} D$$

$$= \frac{-F_0}{2R} 2R\sin\frac{\alpha}{2}$$

$$F = -F_0\sin\frac{\alpha}{2}$$

$$W = \int_c \vec{F}\cdot d\vec{r}$$

$$= \int F\cos\theta\, ds$$

$$= -F_0\int_0^{\pi} \sin\frac{\alpha}{2}\cos\left(\pi - \frac{\alpha}{2}\right) R\,d\alpha$$

$$= +F_0 R\int_0^{\pi} \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\,d\alpha$$

$$\int \sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\,d\alpha = 2\int u\,du = u^2 + C$$

$$u = \sin\frac{\alpha}{2}$$

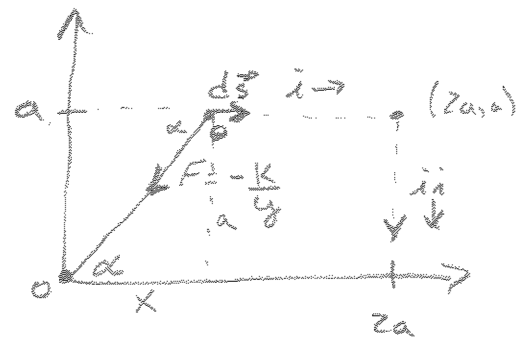
$$du = \frac{1}{2}\cos\frac{\alpha}{2}\,d\alpha$$

$$2\,du = \cos\frac{\alpha}{2}\,d\alpha$$

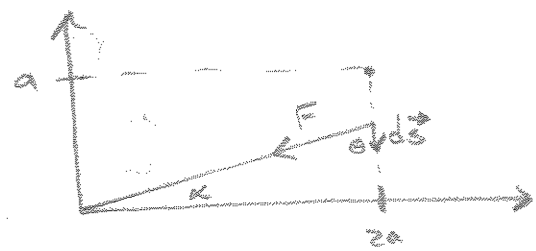
$$= F_0 R\left[\sin^2\frac{\alpha}{2}\right]\Big|_0^{\pi}$$

$$\boxed{W = F_0 R}$$

3-13 a)



$\theta = \pi - \alpha \quad 0 \le x \le 2a,\ y = a$



$\theta = \frac{\pi}{2} - \alpha \quad a \ge y \ge 0,\ x = 2a$

$F = \frac{K}{y}$

i) $W = \int \vec{F} \cdot d\vec{s} = \int |\vec{F}| \cos\theta \, ds$

$= -K \int \frac{\cos\alpha}{y} ds \qquad \cos\theta = \cos(\pi - \alpha) = \cos\pi\cos\alpha + \sin\pi\sin\alpha$

$\qquad = -\cos\alpha$

$= -K \int_0^{2a} \frac{x\,dx}{y(a^2+x^2)^{1/2}} \quad \{y = a\} \qquad \cos\alpha = \frac{x}{(a^2+x^2)^{1/2}}$

$= \frac{-K}{a} \int_0^{2a} \frac{x\,dx}{(a^2+x^2)^{1/2}} \qquad \int \frac{x\,dx}{(a^2+x^2)^{1/2}} = \frac{1}{2}\int \frac{du}{u^{1/2}} \qquad -\cos\alpha = \frac{-x}{(a^2+x^2)^{1/2}}$

$u = a^2 + x^2$

$du = 2x\,dx \qquad = \frac{1}{2}\int u^{-1/2} du$

$\frac{du}{2} = x\,dx \qquad = u^{1/2} + C$

$= \frac{-K}{a} (a^2+x^2)^{1/2} \Big|_0^{2a}$

$= \frac{-K}{a} \left[(a^2 + 4a^2)^{1/2} - a \right]$

$= \frac{-K}{a} \left[\sqrt{5}a - a \right]$

$= -K(\sqrt{5} - 1)$

$= K(1 - \sqrt{5})$

ii) $W = \int \vec{F} \cdot d\vec{s}$

$= \int F \cos\theta \, dy$

$= \int F \sin\alpha \, dy$

$= -K \int_a^0 \frac{dy}{(4a^2+y^2)^{1/2}}$

$= -K \ln \left| \frac{\sqrt{4a^2+y^2}}{2a} + \frac{y}{2a} \right| \Big|_a^0$

$= -K \left[\ln|1| - \ln \left| \frac{\sqrt{5}a + a}{2a} \right| \right.$

$= K \ln \left(\frac{\sqrt{5}a + a}{2a} \right)$

$= K \ln \frac{1}{2}(1+\sqrt{5})$

$\cos\theta = \cos(\frac{\pi}{2} - \alpha)$

$= \cos\frac{\pi}{2}\cos\alpha + \sin\frac{\pi}{2}\sin\alpha$

$= \sin\alpha$

$\sin\alpha = \frac{y}{(4a^2+y^2)^{1/2}}$

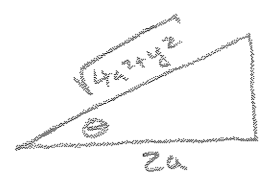
$\int \frac{dy}{(4a^2+y^2)^{1/2}} = \int \frac{2a\sec^2\theta \, d\theta}{(4a^2+4a^2\tan^2\theta)^{1/2}}$

$y^2 = 4a^2\tan^2\theta$ $\quad = \int \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^{1/2}}$

$y = 2a\tan\theta$

$dy = 2a\sec^2\theta \, d\theta$ $\quad = \int \sec\theta \, d\theta$

$= \ln|\sec\theta + \tan\theta| + C$



$\tan\theta = \frac{y}{2a}$

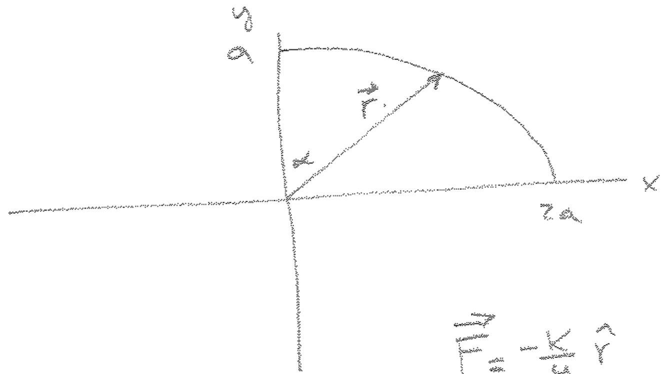
$\sec\theta = \frac{\sqrt{4a^2+y^2}}{2a}$

$= \ln \left| \frac{\sqrt{4a^2+y^2}}{2a} + \frac{y}{2a} \right| + C$

$W = W_i + W_{ii}$

$$\boxed{W = K\left[1 - \sqrt{5} + \ln \frac{1}{2}(1+\sqrt{5})\right]}$$

b)



$$\vec{r} = 2a\sin\alpha\,\hat{x} + a\cos\alpha\,\hat{y}$$

$$\vec{F} = -\frac{k}{y}\hat{r} \qquad \hat{r} = \frac{\vec{r}}{|\vec{r}|}$$

$$= \frac{2a\sin\alpha\,\hat{x} + a\cos\alpha\,\hat{y}}{(4a^2\sin^2\alpha + a^2\cos^2\alpha)^{1/2}}$$

$$= \frac{2\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}}{(4\sin^2\alpha + \cos^2\alpha)^{1/2}}$$

$$\vec{F} = \frac{-k}{a\cos\alpha}\,\frac{2\sin\alpha\,\hat{x} + \cos\alpha\,\hat{y}}{(4\sin^2\alpha + \cos^2\alpha)^{1/2}}$$

$$\frac{d\vec{r}}{d\alpha} = 2a\cos\alpha\,\hat{x} - a\sin\alpha\,\hat{y}$$

$$W = \int \vec{F}\cdot\left(\frac{d\vec{r}}{d\alpha}\right)d\alpha$$

$$= \frac{-k}{a}\int_0^{\pi/2} \frac{2\sin\alpha\, 2a\cos\alpha - a\sin\alpha\cos\alpha}{\cos\alpha(4\sin^2\alpha + \cos^2\alpha)^{1/2}}\,d\alpha$$

$$= \frac{-k}{a}\int_0^{\pi/2} \frac{3a\sin\alpha}{(4\sin^2\alpha + \cos^2\alpha)^{1/2}}\,d\alpha$$

$$= -3k\int_0^{\pi/2} \frac{\sin\alpha\, d\alpha}{(4\sin^2\alpha + \cos^2\alpha)^{1/2}}$$

$$= -3k\int_0^{\pi/2} \frac{\sin\alpha\, d\alpha}{(4(1-\cos^2\alpha) + \cos^2\alpha)^{1/2}}$$

$$= -3k\int_0^{\pi/2} \frac{\sin\alpha\, d\alpha}{(4 - 3\cos^2\alpha)^{1/2}}$$

$$= -3k\int_0^{\pi/2} \frac{\sin\alpha\, d\alpha}{\sqrt{3}(\frac{4}{3} - \cos^2\alpha)^{1/2}}$$

$$= \frac{-3k}{\sqrt{3}}\int_0^{\pi/2} \frac{\sin\alpha\, d\alpha}{(\frac{4}{3} - \cos^2\alpha)^{1/2}}$$

$$\frac{4}{3}u^2 = \cos^2\alpha$$

$$\frac{2}{\sqrt{3}}u = \cos\alpha$$

$$\frac{2}{\sqrt{3}}du = -\sin\alpha\, d\alpha$$

$$-\frac{2}{\sqrt{3}}du = \sin\alpha\, d\alpha$$

$$-\frac{2}{\sqrt{3}}\int \frac{du}{(\frac{4}{3} - \frac{4}{3}u^2)^{1/2}}$$

$$-\frac{2}{\sqrt{3}}\int \frac{du}{\frac{2}{\sqrt{3}}(1-u^2)^{1/2}}$$

$$-\int \frac{du}{(1-u^2)^{1/2}}$$

$$\omega = \frac{3K}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}}{2}\cos\alpha\right)\Big|_0^{\pi/2}$$

$$= \frac{3K}{\sqrt{3}}\left(-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)$$

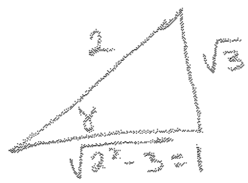
$$= \frac{-3K}{\sqrt{3}}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{-3K}{\sqrt{3}}\left(\frac{\pi}{3}\right)$$

$$\boxed{\omega = \frac{-\pi K}{\sqrt{3}}}$$

$\gamma = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$

$\sin\gamma = \frac{\sqrt{3}}{2}$



$\cos\gamma = \frac{1}{2}$

$\gamma = \frac{\pi}{3}$

$$-\int \frac{du}{(1-u^2)^{1/2}} = -\int \frac{\cos\beta\, d\beta}{(1-\sin^2\beta)^{1/2}}$$

$u = \sin\beta$

$du = \cos\beta\, d\beta$

$$= -\int d\beta$$

$$= -\beta + C$$

$$= -\sin^{-1}(u) + C$$

$$= -\sin^{-1}\left(\frac{\sqrt{3}}{2}\cos\alpha\right) + C$$

3-15

$$\vec{A} = A_r\hat{r} + A_\theta\hat{\theta} + A_z\hat{z}$$

$$\frac{d\hat{r}}{d\theta} = \hat{\theta}, \quad \frac{d\hat{\theta}}{d\theta} = -\hat{r}$$

$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\hat{\theta}$$

$$\frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\hat{r}$$

$$\frac{d\vec{A}}{dt} = \frac{dA_r}{dt}\hat{r} + A_r\frac{d\hat{r}}{dt} + \frac{dA_\theta}{dt}\hat{\theta} + A_\theta\frac{d\hat{\theta}}{dt} + \frac{dA_z}{dt}\hat{z} + A_z\frac{d\hat{z}}{dt}$$

$$= \frac{dA_r}{dt}\hat{r} + A_r\dot{\theta}\hat{\theta} + \frac{dA_\theta}{dt}\hat{\theta} - A_\theta\dot{\theta}\hat{r} + \frac{dA_z}{dt}\hat{z} + A_z\frac{d\hat{z}}{dt}$$

$$= \left(\frac{dA_r}{dt} - A_\theta\dot{\theta}\right)\hat{r} + \left(A_r\dot{\theta} + \frac{dA_\theta}{dt}\right)\hat{\theta} + \frac{dA_z}{dt}\hat{z} + A_z\frac{d\hat{z}}{dt}$$

$$\frac{d^2\vec{A}}{dt^2} = \left(\frac{dA_r}{dt} - A_\theta\dot{\theta}\right)\frac{d\hat{r}}{dt} + \left(\frac{d^2A_r}{dt^2} - \frac{dA_\theta}{dt}\dot{\theta} - A_\theta\ddot{\theta}\right)\hat{r} + \left(A_r\dot{\theta} + \frac{dA_\theta}{dt}\right)\frac{d\hat{\theta}}{dt} +$$

$$\left(\frac{dA_r}{dt}\dot{\theta} + A_r\ddot{\theta} + \frac{d^2A_\theta}{dt^2}\right)\hat{\theta} + \frac{d^2A_z}{dt^2}\hat{z} + \frac{dA_z}{dt}\frac{d\hat{z}}{dt} + \frac{dA_z}{dt}\frac{d\hat{z}}{dt} + A_z\frac{d^2\hat{z}}{dt^2}$$

$$= \left(\dot{A}_r\dot{\theta} - A_\theta\dot{\theta}^2\right)\hat{\theta} + \left(\ddot{A}_r - \dot{A}_\theta\dot{\theta} - A_\theta\ddot{\theta}\right)\hat{r} - \left(A_r\dot{\theta}^2 + \dot{A}_\theta\dot{\theta}\right)\hat{r} +$$

$$\left(\dot{A}_r\dot{\theta} + A_r\ddot{\theta} + \ddot{A}_\theta\right)\hat{\theta} + \ddot{A}_z\hat{z} + 2\dot{A}_z\frac{d\hat{z}}{dt} + A_z\frac{d^2\hat{z}}{dt^2}$$

$$= \left(\ddot{A}_r - \dot{A}_\theta\dot{\theta} - A_\theta\ddot{\theta} - A_r\dot{\theta}^2 - \dot{A}_\theta\dot{\theta}\right)\hat{r} + \left(\dot{A}_r\dot{\theta} - A_\theta\dot{\theta}^2 + \dot{A}_r\dot{\theta} + A_r\ddot{\theta} + \ddot{A}_\theta\right)\hat{\theta} +$$

$$\ddot{A}_z\hat{z} + 2\dot{A}_z\frac{d\hat{z}}{dt} + A_z\frac{d^2\hat{z}}{dt^2}$$

$$\boxed{\frac{d^2\vec{A}}{dt^2} = \left(\ddot{A}_r - 2\dot{A}_\theta\dot{\theta} - A_\theta\ddot{\theta} - A_r\dot{\theta}^2\right)\hat{r} + \left(\ddot{A}_\theta + 2\dot{A}_r\dot{\theta} + A_r\ddot{\theta} - A_\theta\dot{\theta}^2\right)\hat{\theta} + \ddot{A}_z\hat{z} \qquad \left\{\frac{d\hat{z}}{dt} = \frac{d^2\hat{z}}{dt^2} = 0\right\}}$$

3-17

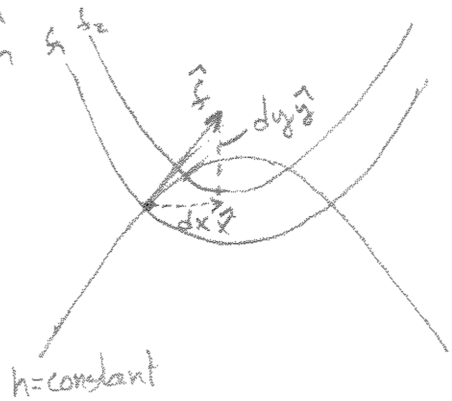
a) $x = f - h \qquad y = 2(fh)^{1/2}$

$$y^2 = 4fh$$
$$h = \frac{y^2}{4f}$$
$$x = f - \frac{y^2}{4f}$$
$$4xf = 4f^2 - y^2$$
$$4f^2 - 4xf = y^2$$
$$f^2 - xf = \frac{y^2}{4}$$
$$f^2 - xf + \frac{x^2}{4} = \frac{y^2+x^2}{4}$$
$$\left(f - \frac{x}{2}\right)^2 = \frac{y^2+x^2}{4}$$
$$f - \frac{x}{2} = \frac{\pm(y^2+x^2)^{1/2}}{2}$$

$$y^2 = 4fh$$
$$f = \frac{y^2}{4h}$$
$$x = \frac{y^2}{4h} - h$$
$$4xh = y^2 - 4h^2$$
$$4h^2 + 4xh = y^2$$
$$h^2 + xh = \frac{y^2}{4}$$
$$h^2 + xh + \frac{x^2}{4} = \frac{y^2+x^2}{4}$$
$$\left(h + \frac{x}{2}\right)^2 = \frac{y^2+x^2}{4}$$
$$h + \frac{x}{2} = \frac{\pm(y^2+x^2)^{1/2}}{2}$$

$$\boxed{f = \frac{x \pm (x^2+y^2)^{1/2}}{2} \qquad h = \frac{-x \pm (x^2+y^2)^{1/2}}{2}}$$

Show $\hat{f} \perp \hat{h}$



$$\hat{f} = \frac{(dx\,\hat{x} + dy\,\hat{y})}{((dx)^2 + (dy)^2)^{1/2}}$$

$\hat{f})$ $\quad x = f - h \;,\; df > 0 \;,\; dh = 0$

$$dx = df - dh \quad \{dh = 0 \;,\; h = \text{constant}\}$$

$$dx = df$$

$$y = 2(fh)^{1/2}$$

$$dy = (fh)^{-1/2}\left[f\,dh + h\,df\right]$$

$$dy = \frac{h\,df}{(fh)^{1/2}}$$

$$\hat{f} = \frac{dx\,\hat{x} + dy\,\hat{y}}{\left[(df)^2 + \frac{(h\,df)^2}{fh}\right]^{1/2}}$$

$$= \frac{df\,\hat{x} + \frac{h\,df}{(fh)^{1/2}}\,\hat{y}}{D_1}$$

$$\hat{f}\cdot\hat{h} = \frac{df}{D_1}\cdot\frac{-dh}{D_2} + \frac{\frac{h\,df}{(fh)^{1/2}}\cdot\frac{f\,dh}{(fh)^{1/2}}}{D_1 D_2}$$

$$= \frac{-df\,dh + \frac{h\,df\cdot f\,dh}{fh}}{D_1 D_2}$$

$$= \frac{-df\,dh + df\,dh}{D_1 D_2}$$

$$\hat{f}\cdot\hat{h} = 0 \Rightarrow \hat{f} \perp \hat{h}$$

$\hat{h})$ $\quad x = f - h \;,\; dh > 0 \;,\; df = 0$

$$dx = df - dh$$

$$dx = -dh$$

$$dy = (fh)^{1/2}\left[f\,dh + h\,df\right]$$

$$dy = \frac{f\,dh}{(fh)^{1/2}}$$

$$\hat{h} = \frac{dx\,\hat{x} + dy\,\hat{y}}{\left[(dh)^2 + \frac{(f\,dh)^2}{fh}\right]^{1/2}}$$

$$= \frac{-dh\,\hat{x} + \frac{f\,dh}{(fh)^{1/2}}\,\hat{y}}{D_2}$$

b) Express $\hat{f}$ and $\hat{h}$ in terms of f and h.

$$x = f - h \qquad y = 2(fh)^{1/2}$$

The unit tangent vector is

$$\hat{T} = \frac{d\vec{R}}{ds} = \frac{dx}{ds}\hat{x} + \frac{dy}{ds}\hat{y} \qquad ds^2 = dx^2 + dy^2$$

$\hat{f} = \hat{T}_f$ when $df > 0$ and $h = $ constant or $dh = 0$

$$dx = df \quad , \quad dy = \frac{h\,df}{(fh)^{1/2}}$$

$$ds^2 = df^2 + \frac{h^2 df^2}{fh}$$

$$= df^2\left(1 + \frac{h}{f}\right)$$

$$ds = df\left(\frac{f+h}{f}\right)^{1/2}$$

$$\hat{f} = \frac{df}{df\left(\frac{f+h}{f}\right)^{1/2}}\hat{x} + \frac{h\,df}{(fh)^{1/2} df\left(\frac{f+h}{f}\right)^{1/2}}\hat{y}$$

$$\boxed{\hat{f} = \left(\frac{f}{f+h}\right)^{1/2}\hat{x} + \left(\frac{h}{f+h}\right)^{1/2}\hat{y}}$$

$\hat{h} = \hat{T}_h$ when $dh > 0$ and $f = $ constant or $df = 0$

$$dx = -dh \; , \; dy = \frac{f\,dh}{(fh)^{1/2}}$$

$$ds^2 = dh^2 + \frac{f^2 dh^2}{fh}$$

$$ds^2 = dh^2\left(\frac{f+h}{h}\right)$$

$$ds = dh\left(\frac{f+h}{h}\right)^{1/2}$$

$$\hat{h} = \frac{-dh}{dh\left(\frac{f+h}{h}\right)^{1/2}}\hat{x} + \frac{f\,dh}{(fh)^{1/2} dh\left(\frac{f+h}{h}\right)^{1/2}}\hat{y}$$

$$\boxed{\hat{h} = -\left(\frac{h}{f+h}\right)^{1/2}\hat{x} + \left(\frac{f}{f+h}\right)^{1/2}\hat{y}}$$

$$\frac{\partial \hat{f}}{\partial f} = \frac{1}{2}\left(\frac{f}{f+h}\right)^{-1/2}\left(\frac{f+h-f}{(f+h)^2}\right)\hat{x} + \frac{1}{2}\left(\frac{h}{f+h}\right)^{-1/2}\left(\frac{0-h}{(f+h)^2}\right)\hat{y}$$

$$= \frac{1}{2}\left(\frac{f+h}{f}\right)^{1/2}\left(\frac{h}{(f+h)^2}\right)\hat{x} - \frac{1}{2}\left(\frac{f+h}{h}\right)^{1/2}\left(\frac{h}{(f+h)^2}\right)\hat{y}$$

$$= -\frac{1}{2}\frac{1}{f+h}\left[-\frac{1}{f^{1/2}}\left(\frac{h}{(f+h)^{1/2}}\right)\hat{x} + \frac{h^{1/2}}{(f+h)^{1/2}}\hat{y}\right]$$

$$= -\frac{1}{2}\frac{1}{f+h}\left[\left(\frac{h}{f}\right)^{1/2}\frac{-h^{1/2}}{(f+h)^{1/2}}\hat{x} + \left(\frac{h}{f}\right)^{1/2}\frac{f^{1/2}}{(f+h)^{1/2}}\hat{y}\right]$$

$$= -\frac{1}{2}\left(\frac{h}{f}\right)^{1/2}\frac{1}{f+h}\left[-\left(\frac{h}{f+h}\right)^{1/2}\hat{x} + \left(\frac{f}{f+h}\right)^{1/2}\hat{y}\right]$$

$$\boxed{\frac{\partial \hat{f}}{\partial f} = -\frac{1}{2}\left(\frac{h}{f}\right)^{1/2}\frac{\hat{h}}{f+h}}$$

$$\frac{\partial \hat{f}}{\partial h} = \frac{1}{2}\left(\frac{f}{f+h}\right)^{-1/2}\left(\frac{-f}{(f+h)^2}\right)\hat{x} + \frac{1}{2}\left(\frac{h}{f+h}\right)^{-1/2}\left(\frac{f+h-h}{(f+h)^2}\right)\hat{y}$$

$$= \frac{1}{2}\left(\frac{f+h}{f}\right)^{1/2}\left(\frac{-f}{(f+h)^2}\right)\hat{x} + \frac{1}{2}\left(\frac{f+h}{h}\right)^{1/2}\left(\frac{f}{(f+h)^2}\right)\hat{y}$$

$$= \frac{1}{2}\frac{1}{f+h}\left[\frac{-f^{1/2}}{(f+h)^{1/2}}\hat{x} + \frac{1}{h^{1/2}}\frac{f}{(f+h)^{1/2}}\hat{y}\right]$$

$$= \frac{1}{2}\frac{1}{f+h}\left[\frac{h^{1/2}}{h^{1/2}}\frac{-f^{1/2}}{(f+h)^{1/2}}\hat{x} + \frac{f^{1/2}}{h^{1/2}}\frac{f^{1/2}}{(f+h)^{1/2}}\hat{y}\right]$$

$$= \frac{1}{2}\left(\frac{f}{h}\right)^{1/2}\frac{1}{f+h}\left[\frac{-h^{1/2}}{(f+h)^{1/2}}\hat{x} + \frac{f^{1/2}}{(f+h)^{1/2}}\hat{y}\right]$$

$$\boxed{\frac{\partial \hat{f}}{\partial h} = \frac{1}{2}\left(\frac{f}{h}\right)^{1/2}\frac{\hat{h}}{f+h}}$$

$$\frac{\partial \hat{h}}{\partial f} = -\frac{1}{2}\left(\frac{h}{f+h}\right)^{-\frac{1}{2}}\left(\frac{-h}{(f+h)^2}\right)\hat{x} + \frac{1}{2}\left(\frac{f}{f+h}\right)^{-\frac{1}{2}}\left(\frac{f+h-f}{(f+h)^2}\right)\hat{y}$$

$$= \frac{1}{2}\left(\frac{f+h}{h}\right)^{1/2}\left(\frac{h}{(f+h)^2}\right)\hat{x} + \frac{1}{2}\left(\frac{f+h}{f}\right)^{1/2}\left(\frac{h}{(f+h)^2}\right)\hat{y}$$

$$= \frac{1}{2}\frac{1}{f+h}\left[\frac{h^{1/2}}{(f+h)^{1/2}}\hat{x} + \frac{1}{f^{1/2}}\frac{h}{(f+h)^{1/2}}\hat{y}\right]$$

$$= \frac{1}{2}\frac{1}{f+h}\left[\frac{f^{1/2}}{f^{1/2}}\frac{h^{1/2}}{(f+h)^{1/2}}\hat{x} + \frac{h^{1/2}}{f^{1/2}}\frac{h^{1/2}}{(f+h)^{1/2}}\hat{y}\right]$$

$$= \frac{1}{2}\left(\frac{h}{f}\right)^{1/2}\frac{1}{f+h}\left[\frac{f^{1/2}}{(f+h)^{1/2}}\hat{x} + \frac{h^{1/2}}{(f+h)^{1/2}}\hat{y}\right]$$

$$\boxed{\frac{\partial \hat{h}}{\partial f} = \frac{1}{2}\left(\frac{h}{f}\right)^{1/2}\frac{\hat{f}}{f+h}}$$

$$\frac{\partial \hat{h}}{\partial h} = -\frac{1}{2}\left(\frac{h}{f+h}\right)^{-\frac{1}{2}}\left(\frac{f+h-h}{(f+h)^2}\right)\hat{x} + \frac{1}{2}\left(\frac{f}{f+h}\right)^{-\frac{1}{2}}\left(\frac{-f}{(f+h)^2}\right)\hat{y}$$

$$= -\frac{1}{2}\left(\frac{f+h}{h}\right)^{1/2}\left(\frac{f}{(f+h)^2}\right)\hat{x} - \frac{1}{2}\left(\frac{f+h}{f}\right)^{1/2}\left(\frac{f}{(f+h)^2}\right)\hat{y}$$

$$= -\frac{1}{2}\frac{1}{f+h}\left[\frac{1}{h^{1/2}}\frac{f}{(f+h)^{1/2}}\hat{x} + \frac{f^{1/2}}{(f+h)^{1/2}}\hat{y}\right]$$

$$= -\frac{1}{2}\frac{1}{f+h}\left[\frac{f^{1/2}}{h^{1/2}}\frac{f^{1/2}}{(f+h)^{1/2}}\hat{x} + \frac{f^{1/2}}{h^{1/2}}\frac{h^{1/2}}{(f+h)^{1/2}}\hat{y}\right]$$

$$\boxed{\frac{\partial \hat{h}}{\partial h} = -\frac{1}{2}\left(\frac{f}{h}\right)^{1/2}\frac{\hat{f}}{f+h}}$$

$$\vec{r} = x\hat{x} + y\hat{y} \qquad x = f - h \quad y = 2(fh)^{1/2}$$

$$\hat{f} = \left(\frac{f}{f+h}\right)^{1/2}\hat{x} + \left(\frac{h}{f+h}\right)^{1/2}\hat{y}$$

$$\hat{h} = -\left(\frac{h}{f+h}\right)^{1/2}\hat{x} + \left(\frac{f}{f+h}\right)^{1/2}\hat{y}$$

$$(f+h)^{1/2}\hat{f} = f^{1/2}\hat{x} + h^{1/2}\hat{y}$$

$$(f+h)^{1/2}\hat{h} = -h^{1/2}\hat{x} + f^{1/2}\hat{y}$$

$$h^{1/2}(f+h)^{1/2}\hat{f} = h^{1/2}f^{1/2}\hat{x} + h\hat{y}$$

$$f^{1/2}(f+h)^{1/2}\hat{h} = -h^{1/2}f^{1/2}\hat{x} + f\hat{y}$$

$$(f+h)^{1/2}(h^{1/2}\hat{f} + f^{1/2}\hat{h}) = (f+h)\hat{y}$$

$$\boxed{\hat{y} = \frac{h^{1/2}\hat{f} + f^{1/2}\hat{h}}{(f+h)^{1/2}}}$$

$$f^{1/2}(f+h)^{1/2}\hat{f} = f\hat{x} + f^{1/2}h^{1/2}\hat{y}$$

$$-h^{1/2}(f+h)^{1/2}\hat{h} = h\hat{x} - f^{1/2}h^{1/2}\hat{y}$$

$$(f+h)^{1/2}(f^{1/2}\hat{f} - h^{1/2}\hat{h}) = (f+h)\hat{x}$$

$$\boxed{\hat{x} = \frac{f^{1/2}\hat{f} - h^{1/2}\hat{h}}{(f+h)^{1/2}}}$$

$$\vec{r} = (f-h)\hat{x} + 2(fh)^{1/2}\hat{y}$$

$$= (f-h)\left[\frac{f^{1/2}\hat{f} - h^{1/2}\hat{h}}{(f+h)^{1/2}}\right] + 2(fh)^{1/2}\left[\frac{h^{1/2}\hat{f} + f^{1/2}\hat{h}}{(f+h)^{1/2}}\right]$$

$$= \frac{(f-h)f^{1/2} + 2(fh)^{1/2}h^{1/2}}{(f+h)^{1/2}}\hat{f} + \frac{-(f-h)h^{1/2} + 2(fh)^{1/2}f^{1/2}}{(f+h)^{1/2}}\hat{h}$$

$$= \frac{(f-h)f^{1/2} + 2f^{1/2}h}{(f+h)^{1/2}}\hat{f} + \frac{-(f-h)h^{1/2} + 2h^{1/2}f}{(f+h)^{1/2}}\hat{h}$$

$$= \frac{f^{1/2}(f-h+2h)}{(f+h)^{1/2}}\hat{f} + \frac{h^{1/2}(h-f+2f)}{(f+h)^{1/2}}\hat{h}$$

$$\boxed{\vec{r} = f^{1/2}(f+h)^{1/2}\hat{f} + h^{1/2}(f+h)^{1/2}\hat{h}}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(x\hat{x} + y\hat{y})$$

$$= \frac{dx}{dt}\hat{x} + x\frac{d\hat{x}}{dt} + \frac{dy}{dt}\hat{y} + y\frac{d\hat{y}}{dt} \qquad \frac{d\hat{x}}{dt} = \frac{d\hat{y}}{dt} = 0$$

$$= \left(\frac{df}{dt} - \frac{dh}{dt}\right)\hat{x} + (fh)^{-1/2}\left(f\frac{dh}{dt} + h\frac{df}{dt}\right)\hat{y}$$

$$= (\dot{f} - \dot{h})\hat{x} + (fh)^{-1/2}(f\dot{h} + h\dot{f})\hat{y}$$

$$= (\dot f - \dot h)\frac{f^{1/2}\hat f - h^{1/2}\hat h}{(f+h)^{1/2}} + (fh)^{-1/2}(f\dot h + h\dot f)\frac{h^{1/2}\hat f + f^{1/2}\hat h}{(f+h)^{1/2}}$$

$$= \left[\frac{(\dot f - \dot h)f^{1/2} + (fh)^{-1/2}(f\dot h + h\dot f)h^{1/2}}{(f+h)^{1/2}}\right]\hat f + \left[\frac{(\dot h - \dot f)h^{1/2} + (fh)^{-1/2}(f\dot h + h\dot f)f^{1/2}}{(f+h)^{1/2}}\right]\hat h$$

$$= \frac{1}{(f+h)^{1/2}}\left(\left[(\dot f-\dot h)f^{1/2} + \frac{(f\dot h + h\dot f)}{f^{1/2}}\right]\hat f + \left[(\dot h - \dot f)h^{1/2} + \frac{(f\dot h + h\dot f)}{h^{1/2}}\right]\hat h\right)$$

$$= \frac{1}{(f+h)^{1/2}}\left(\left[f^{1/2}\dot f - f^{1/2}\dot h + f^{1/2}\dot h + \frac{h}{f^{1/2}}\dot f\right]\hat f + \left[h^{1/2}\dot h - h^{1/2}\dot f + \frac{f}{h^{1/2}}\dot h + h^{1/2}\dot f\right]\hat h\right)$$

$$= \frac{1}{(f+h)^{1/2}}\left(\left[f^{1/2}(\dot f - \dot h + \dot h) + \frac{h}{f^{1/2}}\dot f\right]\hat f + \left[h^{1/2}(\dot h - \dot f + \dot f) + \frac{f}{h^{1/2}}\dot h\right]\hat h\right)$$

$$= \frac{1}{(f+h)^{1/2}}\left(\left[f^{1/2}\dot f + \frac{h}{f^{1/2}}\dot f\right]\hat f + \left[h^{1/2}\dot h + \frac{f}{h^{1/2}}\dot h\right]\hat h\right)$$

$$= \frac{1}{(f+h)^{1/2}}\left(\dot f\left(f^{1/2} + \frac{h}{f^{1/2}}\right)\hat f + \dot h\left(h^{1/2} + \frac{f}{h^{1/2}}\right)\hat h\right)$$

$$= \frac{1}{(f+h)^{1/2}}\left(\dot f\left(\frac{f+h}{f^{1/2}}\right)\hat f + \dot h\left(\frac{h+f}{h^{1/2}}\right)\hat h\right)$$

$$= \frac{f+h}{(f+h)^{1/2}}\left(\frac{\dot f}{f^{1/2}}\hat f + \frac{\dot h}{h^{1/2}}\hat h\right)$$

$$\boxed{\dot{\vec r} = (f+h)^{1/2}\left(\frac{\dot f}{f^{1/2}}\hat f + \frac{\dot h}{h^{1/2}}\hat h\right)}$$

$$\ddot{\vec r} = \vec a = \frac{d\vec v}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\hat x + x\frac{d\hat x}{dt} + \frac{dy}{dt}\hat y + y\frac{d\hat y}{dt}\right)$$

$$= \frac{d}{dt}\left(\frac{dx}{dt}\hat x + \frac{dy}{dt}\hat y\right) \quad \left\{\frac{d\hat x}{dt} = \frac{d\hat y}{dt} = 0\right\}$$

$$= \frac{d^2x}{dt^2}\hat x + \frac{dx}{dt}\frac{d\hat x}{dt} + \frac{d^2y}{dt^2}\hat y + \frac{dy}{dt}\frac{d\hat y}{dt}$$

$$= \frac{d^2x}{dt^2}\hat x + \frac{d^2y}{dt^2}\hat y$$

$$x = f - h \qquad y = 2(fh)^{1/2}$$

$$\frac{dx}{dt} = \dot f - \dot h \qquad \frac{dy}{dt} = (fh)^{-1/2}(f\dot h + h\dot f)$$

$$\frac{d^2x}{dt^2} = \ddot f - \ddot h \qquad \frac{d^2y}{dt^2} = (fh)^{-1/2}\left[\dot f\dot h + f\ddot h + h\ddot f + \dot h\dot f\right] + (f\dot h + h\dot f)\left(-\tfrac{1}{2}\right)(fh)^{-3/2}(f\dot h + h\dot f)$$

$$= (fh)^{-1/2}\left[f\ddot h + \dot f\dot h + h\ddot f + \dot h\dot f\right] - \tfrac{1}{2}(f\dot h + h\dot f)^2(fh)^{-3/2}$$

$$= (fh)^{-1/2}\left[f\ddot h + 2\dot f\dot h + h\ddot f\right] - \tfrac{1}{2}(f\dot h + h\dot f)^2(fh)^{-3/2}$$

$$\vec a = (\ddot f - \ddot h)\hat x + \left[(fh)^{-1/2}(f\ddot h + 2\dot f\dot h + h\ddot f) - \tfrac{1}{2}(f\dot h + h\dot f)^2(fh)^{-3/2}\right]\hat y$$

$$(\ddot f - \ddot h)\hat x = (\ddot f - \ddot h)\frac{f^{1/2}\hat f - h^{1/2}\hat h}{(f+h)^{1/2}}$$

$$= \frac{(\ddot f - \ddot h)f^{1/2}}{(f+h)^{1/2}}\hat f - \frac{(\ddot f - \ddot h)h^{1/2}}{(f+h)^{1/2}}\hat h$$

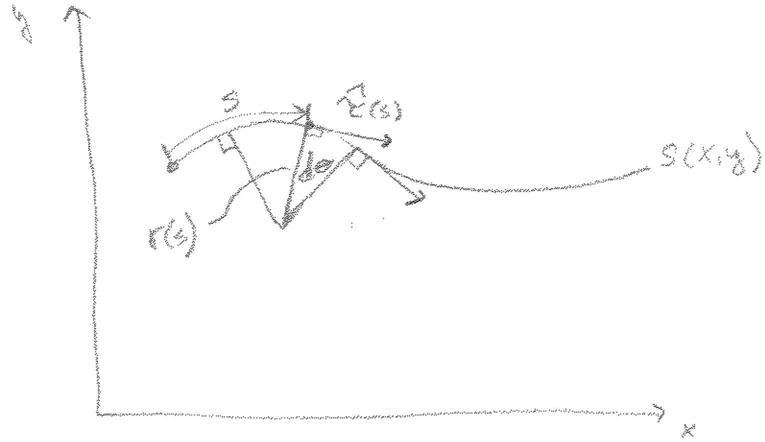
$\hat y$ term: $$\left[(fh)^{-1/2}(f\ddot h + 2\dot f\dot h + h\ddot f) - \tfrac{1}{2}(f\dot h + h\dot f)^2(fh)^{-3/2}\right]\frac{h^{1/2}\hat f + f^{1/2}\hat h}{(f+h)^{1/2}} =$$

$$\frac{\left[f^{-1/2}(f\ddot h + 2\dot f\dot h + h\ddot f) - \dfrac{h^{1/2}(f\dot h + h\dot f)^2}{2f^{3/2}h^{3/2}}\right]}{(f+h)^{1/2}}\hat f +$$

$$\frac{\left[h^{-1/2}(f\ddot h + 2\dot f\dot h + h\ddot f) - \dfrac{f^{1/2}(f\dot h + h\dot f)^2}{2f^{3/2}h^{3/2}}\right]}{(f+h)^{1/2}}\hat h$$

$$\boxed{\ddot{\vec r} = \left[\left[(\ddot f - \ddot h)f^{1/2} + f^{-1/2}(f\ddot h + 2\dot h\dot f + h\ddot f) - \frac{(f\dot h + h\dot f)^2}{2f^{3/2}h}\right]\hat f - \left[(\ddot f - \ddot h)h^{1/2} - h^{-1/2}(f\ddot h + 2\dot f\dot h + h\ddot f) + \frac{(f\dot h + h\dot f)^2}{2h^{3/2}f}\right]\hat h\right]\cdot\frac{1}{(f+h)^{1/2}}}$$

3-19



$$r\,d\theta = ds$$

$$r\frac{d\theta}{dt} = \frac{ds}{dt} = v(s)$$

$$r\omega = v(s)$$

$$v(s)\,\hat{\tau}(s) = \vec{V}(s)$$

$$r\omega\,\hat{\tau}(s) = \vec{V}(s)$$

$$\hat{\tau}(s) = \frac{\vec{V}(s)}{r\omega}$$

$$\frac{d\hat{\tau}}{ds} = \frac{\vec{V}}{r\omega}\frac{1}{ds}$$

$$= \frac{1}{r}\frac{\vec{V}}{\omega ds} \qquad \omega ds = |\vec{V}|$$

$$= \frac{1}{r}\frac{\vec{V}}{|\vec{V}|}$$

$$\boxed{\frac{d\hat{\tau}}{ds} = \frac{\hat{v}}{r}}$$

$$v(s)\,\hat{\tau} = \vec{V} \qquad r\,d\theta = ds$$

$$\boxed{\dot{s}\,\hat{\tau} = \vec{V}} \qquad r\omega = \frac{ds}{dt} = \dot{s} = v(s)$$

$$\vec{a} = \frac{d\vec{V}}{dt}$$

$$= \frac{d}{dt}(\dot{s}\,\hat{\tau})$$

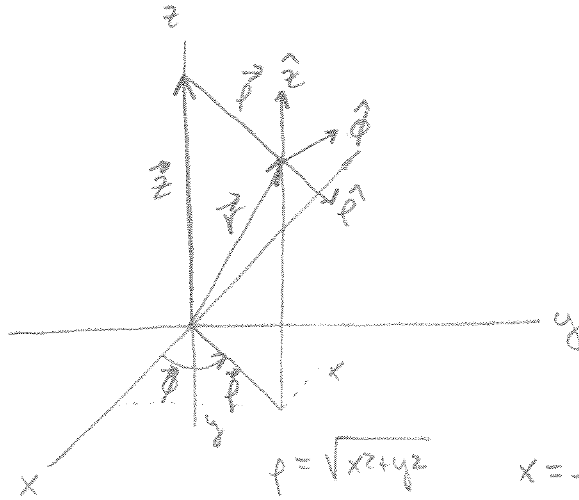
$$= \ddot{s}\,\hat{\tau} + \dot{s}\frac{d\hat{\tau}}{dt}$$

$$\vec{a} = \ddot{s}\hat{t} + \dot{s}\frac{d\hat{t}}{ds}\cdot\frac{ds}{dt}$$

$$= \ddot{s}\hat{t} + \dot{s}^2\frac{d\hat{t}}{ds} \quad \left\{\frac{d\hat{t}}{ds} = \frac{\hat{v}}{r}\right\}$$

$$\boxed{\vec{a} = \ddot{s}\hat{t} + \frac{\dot{s}^2}{r}\hat{v}}$$

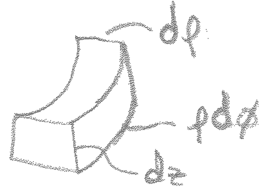
3-21 curl $\vec{A}$ in cylindrical coordinates



$\rho = \sqrt{x^2+y^2}$ $\quad x = \rho\cos\phi \quad y = \rho\sin\phi$

$z = z$

Del operator



$$\vec\nabla = \frac{\partial}{\partial s_1}\hat e_1 + \frac{\partial}{\partial s_2}\hat e_2 + \frac{\partial}{\partial s_3}\hat e_3$$

$ds_1 = d\rho,\ ds_2 = \rho\, d\phi,\ ds_3 = dz,\ \hat e_1 = \hat\rho,\ \hat e_2 = \hat\phi,\ \hat e_3 = \hat z$

$$\vec\nabla = \frac{\partial}{\partial\rho}\hat\rho + \frac{\partial}{\rho\,\partial\phi}\hat\phi + \frac{\partial}{\partial z}\hat z$$

$$\vec A = A_\rho\hat\rho + A_\phi\hat\phi + A_z\hat z$$

$$\vec\nabla\times\vec A = \hat\rho\times\frac{\partial\vec A}{\partial\rho} + \frac{\hat\phi}{\rho}\times\frac{\partial\vec A}{\partial\phi} + \hat z\times\frac{\partial\vec A}{\partial z}$$

$$= \hat\rho\times\left[\frac{\partial A_\rho}{\partial\rho}\hat\rho + \frac{\partial A_\phi}{\partial\rho}\hat\phi + \frac{\partial A_z}{\partial z}\hat z + A_\rho\frac{\partial\hat\rho}{\partial\rho} + A_\phi\frac{\partial\hat\phi}{\partial\rho} + A_z\frac{\partial\hat z}{\partial\rho}\right] +$$

$$\frac{\hat\phi}{\rho}\times\left[\frac{\partial A_\rho}{\partial\phi}\hat\rho + \frac{\partial A_\phi}{\partial\phi}\hat\phi + \frac{\partial A_z}{\partial\phi}\hat z + A_\rho\frac{\partial\hat\rho}{\partial\phi} + A_\phi\frac{\partial\hat\phi}{\partial\phi} + A_z\frac{\partial\hat z}{\partial\phi}\right] +$$

$$\hat z\times\left[\frac{\partial A_\rho}{\partial z}\hat\rho + \frac{\partial A_\phi}{\partial z}\hat\phi + \frac{\partial A_z}{\partial z}\hat z + A_\rho\frac{\partial\hat\rho}{\partial z} + A_\phi\frac{\partial\hat\phi}{\partial z} + A_z\frac{\partial\hat z}{\partial z}\right]$$

$$\hat\rho = \frac{\vec\rho}{\rho} = \frac{x\hat x + y\hat y}{\rho} = \frac{\rho\cos\phi\,\hat x + \rho\sin\phi\,\hat y}{\rho}$$

$$\hat\rho = \cos\phi\,\hat x + \sin\phi\,\hat y$$

$$\hat\phi = \hat z\times\hat\rho = \hat z\times\cos\phi\,\hat x + \hat z\times\sin\phi\,\hat y = \cos\phi\,\hat y - \sin\phi\,\hat x = -\sin\phi\,\hat x + \cos\phi\,\hat y$$

$$\hat z = \hat z$$

The rectangular unit vectors, $\hat x$, $\hat y$, and $\hat z$, are not functions of position. So,

$$\frac{\partial\hat\rho}{\partial\rho} = 0,\quad \frac{\partial\hat\phi}{\partial\rho} = 0,\quad \frac{\partial\hat z}{\partial\rho} = 0$$

$$\frac{\partial\hat\rho}{\partial\phi} = -\sin\phi\,\hat x + \cos\phi\,\hat y = \hat\phi$$

$$\frac{\partial\hat\phi}{\partial\phi} = -\cos\phi\,\hat x - \sin\phi\,\hat y = -\hat\rho$$

$$\frac{\partial\hat z}{\partial\phi} = 0$$

$$\frac{\partial\hat\rho}{\partial z} = 0,\quad \frac{\partial\hat\phi}{\partial z} = 0,\quad \frac{\partial\hat z}{\partial z} = 0$$

$$\vec{\nabla}\times\vec{A} = \hat{\rho}\times\left[\frac{\partial A_\rho}{\partial \rho}\hat{\rho} + \frac{\partial A_\phi}{\partial \rho}\hat{\phi} + \frac{\partial A_z}{\partial \rho}\hat{z} + 0+0+0\right] +$$

$$\frac{\hat{\phi}}{\rho}\times\left[\frac{\partial A_\rho}{\partial \phi}\hat{\rho} + \frac{\partial A_\phi}{\partial \phi}\hat{\phi} + \frac{\partial A_z}{\partial \phi}\hat{z} + A_\rho\hat{\phi} - A_\phi\hat{\rho} + 0\right] +$$

$$\hat{z}\times\left[\frac{\partial A_\rho}{\partial z}\hat{\rho} + \frac{\partial A_\phi}{\partial z}\hat{\phi} + \frac{\partial A_z}{\partial z}\hat{z} + 0+0+0\right]$$

$\hat{\rho}\times\hat{\phi}=\hat{z}$ $\quad\hat{\phi}\times\hat{\rho}=-\hat{z}$ $\quad\hat{z}\times\hat{\rho}=\hat{\phi}$

$\hat{\rho}\times\hat{z}=-\hat{\phi}$ $\quad\hat{\phi}\times\hat{z}=\hat{\rho}$ $\quad\hat{z}\times\hat{\phi}=-\hat{\rho}$

$$= \frac{\partial A_\phi}{\partial \rho}\hat{z} - \frac{\partial A_z}{\partial \rho}\hat{\phi} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\hat{z} + \frac{1}{\rho}\frac{\partial A_z}{\partial \phi}\hat{\rho} + \frac{1}{\rho}A_\phi\hat{z} +$$

$$\frac{\partial A_\rho}{\partial z}\hat{\phi} - \frac{\partial A_\phi}{\partial z}\hat{\rho}$$

$$\boxed{\vec{\nabla}\times\vec{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\rho} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right)\hat{\phi} + \left(\frac{\partial A_\phi}{\partial \rho} + \frac{A_\phi}{\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)\hat{z}}$$

Alternate Method:

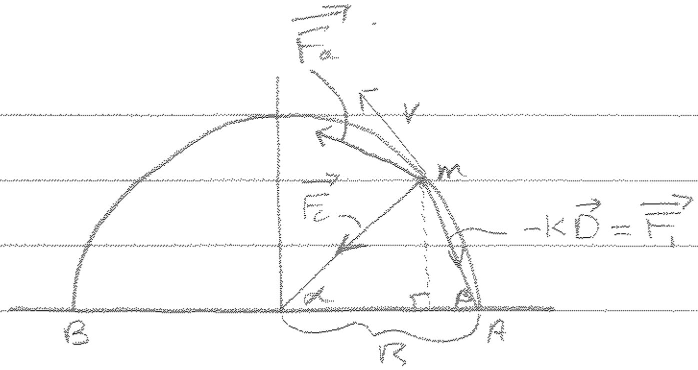
$$\vec{\nabla}\times\vec{A} = \begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial \rho} & \frac{1}{\rho}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_\rho & A_\phi & A_z \end{vmatrix}$$

$$= \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{z}\left(\frac{\partial A_\phi}{\partial \rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)$$

$$= \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{z}\,\frac{1}{\rho}\left(\frac{\partial \rho A_\phi}{\partial \rho} - \frac{\partial A_\rho}{\partial \phi}\right)$$

$$\boxed{\vec{\nabla}\times\vec{A} = \hat{\rho}\left(\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + \hat{\phi}\left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right) + \hat{z}\left(\frac{\partial A_\phi}{\partial \rho} + \frac{A_\phi}{\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial \phi}\right)}$$

3-23



$\vec{F_c} = \vec{F_1} + \vec{F_a}$

$\vec{F_a} = \vec{F_c} - \vec{F_1}$

$\vec{F_c} = -\frac{mv^2}{R}\cos\alpha\,\hat{x} - \frac{mv^2}{R}\sin\alpha\,\hat{y}$

$F_{a_x} = \frac{-mv^2}{R}\cos\alpha - \left(\frac{F_0}{2} - \frac{F_0}{2}\cos\alpha\right)$

$= -\frac{F_0}{2} + \left(\frac{F_0}{2} - \frac{mv^2}{R}\right)\cos\alpha$

$F_{a_y} = -\frac{mv^2}{R}\sin\alpha - \left(-\frac{F_0}{2}\sin\alpha\right)$

$= \left(\frac{F_0}{2} - \frac{mv^2}{R}\right)\sin\alpha$

$$\boxed{\vec{F_a} = \frac{-F_0}{2} + \left(\frac{F_0}{2} - \frac{mv^2}{R}\right)\cos\alpha\,\hat{x} + \left(\frac{F_0}{2} - \frac{mv^2}{R}\right)\sin\alpha\,\hat{y}}$$

$F_1 = -kD$

$F_0 = -k(2R)$

$\frac{F_0}{2R} = -k$

$F_1 = \frac{F_0}{2R}D$

$\vec{F_1} = -\frac{F_0}{2R}\vec{D}$

$D^2 = R^2 + R^2 - 2R^2\cos\alpha$

$= 2R^2 - 2R^2\cos\alpha$

$= 2R^2(1-\cos\alpha)$

$D_y = R\sin\alpha$

$\vec{D_y} = R\sin\alpha\,\hat{y}$

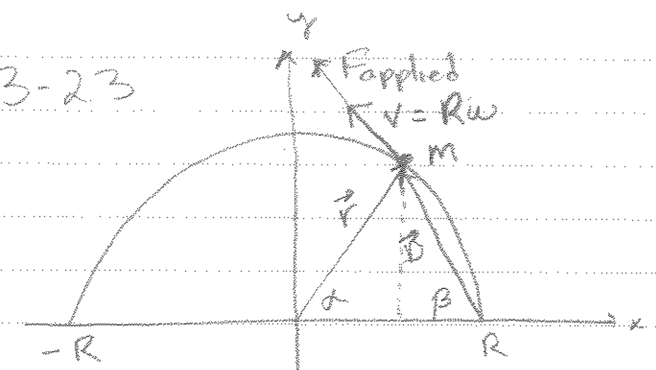
$D_x = R - R\cos\alpha$

$\vec{D_x} = -R(1-\cos\alpha)\,\hat{x}$

$\vec{F_1} = -\frac{F_0}{2R}\left(-R(1-\cos\alpha)\hat{x} + R\sin\alpha\,\hat{y}\right)$

$\vec{F_1} = \left(\frac{F_0}{2} - \frac{F_0}{2}\cos\alpha\right)\hat{x} - \frac{F_0}{2}\sin\alpha\,\hat{y}$

3-23



a) $\vec{F}_c = \frac{-mv^2}{R}\hat{r}$ $\qquad -\hat{r} = -\cos\alpha\,\hat{x} - \sin\alpha\,\hat{y}$

$\vec{F}_1 = -K\vec{D}$ $\qquad D_x = R - R\cos\alpha$

$F_1 = -KD$ $\qquad = R(1-\cos\alpha)$

$F_0 = -K2R$ $\qquad D_y = R\sin\alpha$

$\frac{F_0}{2R} = -K$

$F_{1x} = KD_x$

$= KR(1-\cos\alpha) = \frac{F_0}{2}(1-\cos\alpha)$

$F_{1y} = -KR\sin\alpha = -\frac{F_0}{2}\sin\alpha$

$F_{cx} = -m\frac{v^2}{R}\cos\alpha$

$F_{cy} = -\frac{mv^2}{R}\sin\alpha$ $\qquad F_{ay} = F_{cy} - F_{1y}$

$\vec{F}_c = \vec{F}_1 + \vec{F}_{applied}$ $\qquad = -\frac{mv^2}{R}\sin\alpha + \frac{F_0}{2}\sin\alpha$

$= \vec{F}_1 + \vec{F}_a$ $\qquad \boxed{F_y = \left(\frac{F_0}{2} - \frac{mv^2}{R}\right)\sin\left(\frac{vt}{R}\right)}$

$\vec{F}_a = \vec{F}_c - \vec{F}_1$

$F_x = -F_{ax} = -F_{cx} + F_{1x}$

$= \frac{mv^2}{R}\cos\alpha + \frac{F_0}{2}(1-\cos\alpha)$

$= -\frac{F_0}{2}\cos\alpha + \frac{F_0}{2} + \frac{mv^2}{R}\cos\alpha$ $\qquad v = R\omega = R\frac{d\alpha}{dt}$

$\qquad \frac{vt}{R} = \alpha$

$$\boxed{F_x = \frac{F_0}{2} + \left(\frac{mv^2}{R} - \frac{F_0}{2}\right)\cos\left(\frac{vt}{R}\right)}$$

b) $\vec{I} = \int_{t_1}^{t_2} \vec{F}\,dt$

$\vec{F} = F_x \hat{x} + F_y \hat{y}$

$= \left(\frac{F_0}{2} + \left(\frac{mv^2}{R} - \frac{F_0}{2}\right)\cos\left(\frac{vt}{R}\right)\right)\hat{x} + \left(\frac{F_0}{2} - \frac{mv^2}{R}\right)\sin\left(\frac{vt}{R}\right)$

$\vec{I} = \int_{t_1}^{t_2} \left[\frac{F_0}{2} + \left(\frac{mv^2}{R} - \frac{F_0}{2}\right)\cos\left(\frac{vt}{R}\right)\right]\hat{x}\,dt + \int_{t_1}^{t_2} \left[\left(\frac{F_0}{2} - \frac{mv^2}{R}\right)\sin\left(\frac{vt}{R}\right)\right]\hat{y}\,dt$

$t_1 = 0 \quad t_2 = \frac{\pi R}{v}$

$I_x = \frac{F_0}{2}\Big|_0^{\frac{\pi R}{v}} + \frac{R}{v}\left(\frac{mv^2}{R} - \frac{F_0}{2}\right)\sin\left(\frac{vt}{R}\right)\Big|_0^{\frac{\pi R}{v}}$

$= \frac{\pi F_0 R}{2v} + \frac{R}{v}\left(\frac{mv^2}{R} - \frac{F_0}{2}\right)(0-0)$

$I_x = \frac{\pi F_0 R}{2v}$

$I_y = -\frac{R}{v}\left(\frac{F_0}{2} - \frac{mv^2}{R}\right)\cos\left(\frac{vt}{R}\right)\Big|_0^{\frac{\pi R}{v}}$

$= -\frac{R}{v}\left(\frac{F_0}{2} - \frac{mv^2}{R}\right)(-1-1)$

$= -\frac{R}{v}\left(\frac{F_0}{2} - \frac{mv^2}{R}\right)(-2)$

$I_y = \frac{F_0 R}{v} - 2mv$

$$\boxed{\vec{I} = \frac{\pi F_0 R}{2v}\hat{x} + \left(\frac{F_0 R}{v} - 2mv\right)\hat{y}}$$

3-25

$x = x_0 + at^2$
$y = bt^3$
$z = ct$

$\vec{L} = \vec{r} \times \vec{p}$ $\qquad \vec{r} = (x_0 + at^2)\hat{x} + bt^3\hat{y} + ct\hat{z}$

$= m(\vec{r} \times \vec{v})$ $\qquad \vec{v} = \frac{d\vec{r}}{dt} = 2at\hat{x} + 3bt^2\hat{y} + c\hat{z}$

$$= m \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_0 + at^2 & bt^3 & ct \\ 2at & 3bt^2 & c \end{vmatrix}$$

$$= m\left[(bct^3 - 3bct^3)\hat{x} + (2act^2 - x_0c - act^2)\hat{y} + (3x_0bt^2 + 3abt^4 - 2abt^4)\hat{z}\right]$$

$$\boxed{\vec{L} = m\left[-2bct^3\hat{x} + (act^2 - x_0c)\hat{y} + (3x_0bt^2 + abt^4)\hat{z}\right]}$$

$\vec{F} = m\vec{a}$ $\qquad \vec{a} = \frac{d\vec{v}}{dt} = 2a\hat{x} + 6bt\hat{y}$

$$\boxed{\vec{F} = 2ma\hat{x} + 6mbt\hat{y}}$$

$\vec{N} = \frac{d\vec{L}}{dt} = \vec{r} \times \vec{F}$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_0 + at^2 & bt^3 & ct \\ 2ma & 6mbt & 0 \end{vmatrix}$$

$$= (0 - 6mbct^2)\hat{x} + (2mact - 0)\hat{y} + (6x_0mbt + 6mabt^3 - 2mabt^3)\hat{z}$$

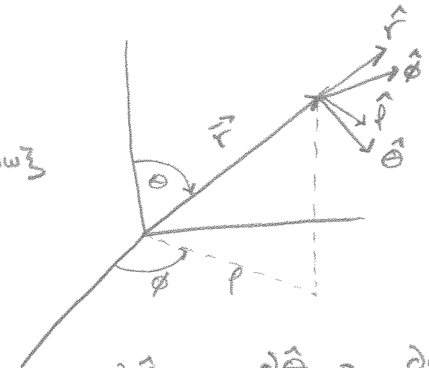
$$\boxed{\vec{N} = -6mbct^2\hat{x} + 2mact\hat{y} + (6x_0mbt + 4abt^3)\hat{z}}$$

$$\frac{dL}{dt} = m\left[-6bct^2\hat{x} + 2act\,\hat{y} + (6x_0bt + 4abt^3)\hat{z}\right]$$

$$\boxed{\vec{N} = \frac{d\vec{L}}{dt}}$$

3-27

$$\vec{r} = r\hat{r}(\theta,\phi)$$



$\hat{\rho} = \hat{r}\sin\theta + \hat{\theta}\cos\theta$
{$\hat{\rho}$ is in the $\hat{r}$-$\hat{\theta}$ plane}

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\,\hat{\phi} \quad \{\text{see } \tfrac{d\hat{r}}{dt} \text{ below}\}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} +$$
$$\dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}\frac{d\hat{\theta}}{dt} +$$
$$\dot{r}\dot{\phi}\sin\theta\,\hat{\phi} + r\ddot{\phi}\sin\theta\,\hat{\phi} +$$
$$r\dot{\phi}\dot{\theta}\cos\theta\,\hat{\phi} + r\dot{\phi}\sin\theta\frac{d\hat{\phi}}{dt}$$

$$\frac{\partial\hat{r}}{\partial r} = 0,\ \frac{\partial\hat{\theta}}{\partial r} = 0,\ \frac{\partial\hat{\phi}}{\partial r} = 0$$
$$\frac{\partial\hat{r}}{\partial\theta} = \hat{\theta},\ \frac{\partial\hat{\theta}}{\partial\theta} = -\hat{r}\ \ \frac{\partial\hat{\phi}}{\partial\theta} = 0$$
$$\frac{\partial\hat{r}}{\partial\phi} = \sin\theta\,\hat{\phi},\ \frac{\partial\hat{\theta}}{\partial\phi} = \cos\theta\,\hat{\phi}$$
$$\frac{\partial\hat{\phi}}{\partial\phi} = -\hat{\rho} = -\sin\theta\,\hat{r} - \cos\theta\,\hat{\theta}$$

$$\frac{d\hat{r}}{dt} = \frac{\partial\hat{r}}{\partial r}\frac{dr}{dt} + \frac{\partial\hat{r}}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\hat{r}}{\partial\phi}\frac{d\phi}{dt}$$
$$= 0 + \dot{\theta}\hat{\theta} + \dot{\phi}\sin\theta\,\hat{\phi}$$

$$\frac{d\hat{\theta}}{dt} = \frac{\partial\hat{\theta}}{\partial r}\frac{dr}{dt} + \frac{\partial\hat{\theta}}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\hat{\theta}}{\partial\phi}\frac{d\phi}{dt}$$
$$= 0 - \dot{\theta}\hat{r} + \dot{\phi}\cos\theta\,\hat{\phi}$$

$$\frac{d\hat{\phi}}{dt} = \frac{\partial\hat{\phi}}{\partial r}\frac{dr}{dt} + \frac{\partial\hat{\phi}}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\hat{\phi}}{\partial\phi}\frac{d\phi}{dt}$$
$$= 0 + 0 - \dot{\phi}\sin\theta\,\hat{r} - \dot{\phi}\cos\theta\,\hat{\theta}$$

$$\vec{a} = \ddot{r}\hat{r} + \dot{r}[\dot{\theta}\hat{\theta} + \dot{\phi}\sin\theta\,\hat{\phi}] + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} + r\dot{\theta}[-\dot{\theta}\hat{r} + \dot{\phi}\cos\theta\,\hat{\phi}] +$$
$$\dot{r}\dot{\phi}\sin\theta\,\hat{\phi} + r\ddot{\phi}\sin\theta\,\hat{\phi} + r\dot{\phi}\dot{\theta}\cos\theta\,\hat{\phi} + r\dot{\phi}\sin\theta[-\dot{\phi}\sin\theta\,\hat{r} - \dot{\phi}\cos\theta\,\hat{\theta}]$$

$$= \ddot{r}\hat{r} - r\dot{\theta}^2\hat{r} - r\dot{\phi}^2\sin\theta\,\hat{r} +$$
$$\dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\,\hat{\theta} +$$
$$\dot{r}\dot{\phi}\sin\theta\,\hat{\phi} + r\dot{\theta}\dot{\phi}\cos\theta\,\hat{\phi} + \dot{r}\dot{\phi}\sin\theta\,\hat{\phi} + r\ddot{\phi}\sin\theta + r\dot{\phi}\dot{\theta}\cos\theta\,\hat{\phi}$$

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin\theta)\hat{r} +$$
$$(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta} +$$
$$(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{\phi}$$

$$\vec{L}_o = \vec{r} \times m\vec{V}$$

$$= r\hat{r} \times m(\dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\dot{\phi}\sin\theta\hat{\phi})$$

$$= r\hat{r} \times mr\dot{\theta}\hat{\theta} + r\hat{r} \times r\dot{\phi}\sin\theta\hat{\phi} \qquad \hat{r}\times\hat{\theta} = \hat{\phi}$$

$$\hat{r}\times\hat{\phi} = -\hat{\theta}$$

$$\vec{L}_o = mr^2\dot{\theta}\hat{\phi} - r^2\dot{\phi}\sin\theta\,\hat{\theta}$$

$$\vec{F}_r = m\vec{a}_r$$
$$= m(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin\theta)\hat{r}$$

$$\vec{F}_\theta = m\vec{a}_\theta$$
$$= m(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta}$$

$$\vec{F}_\phi = m\vec{a}_\phi$$
$$= m(r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{\phi}$$

$$\vec{N} = \vec{r} \times \vec{F}$$
$$= \vec{r} \times (\vec{F}_r + \vec{F}_\theta + \vec{F}_\phi)$$
$$= r\hat{r} \times \vec{F}_r + r\hat{r} \times \vec{F}_\theta + r\hat{r} \times \vec{F}_\phi$$
$$= 0 + m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\dot{\phi}^2\sin\theta\cos\theta)\hat{\phi} - m(r^2\ddot{\phi}\sin\theta + 2r\dot{r}\dot{\phi}\sin\theta + 2r^2\dot{\theta}\dot{\phi}\cos\theta)\hat{\theta}$$

$$\frac{d\vec{L}_o}{dt} = \frac{d}{dt}(\vec{r} \times m\vec{v})$$
$$= \vec{r} \times \frac{d}{dt}(m\vec{v}) + \frac{d\vec{r}}{dt} \times m\vec{v}$$
$$= \vec{r} \times m\frac{d\vec{v}}{dt} + \vec{v} \times m\vec{v}$$
$$= \vec{r} \times m\frac{d\vec{v}}{dt}$$
$$= r\hat{r} \times m\vec{a}$$
$$= r\hat{r} \times m\left[(\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin\theta)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta)\hat{\theta} + (r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\theta}\dot{\phi}\cos\theta)\hat{\phi}\right]$$
$$= 0 + m(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\dot{\phi}^2\sin\theta\cos\theta)\hat{\phi} - m(r^2\ddot{\phi}\sin\theta + 2r\dot{r}\dot{\phi}\sin\theta + 2r^2\dot{\theta}\dot{\phi}\cos\theta)\hat{\theta}$$

$$\frac{d\vec{L}_o}{dt} = \vec{N}$$

3-29

$$x_m = (mV_{x0}/b)(1-\delta) \qquad \delta \ll 1, \quad \frac{bV_{z0}}{mg} \gg 1$$

$$z = \left(\frac{mg}{bV_{x0}} + \frac{V_{z0}}{V_{x0}}\right)x - \frac{m^2g}{b^2}\ln\left(\frac{mV_{x0}}{mV_{x0}-bx}\right)$$

$$0 = \left(\frac{mg}{bV_{x0}} + \frac{V_{z0}}{V_{x0}}\right)\left(\frac{mV_{x0}}{b}\right)(1-\delta) - \frac{m^2g}{b^2}\ln\left(\frac{mV_{x0}}{mV_{x0}-mV_{x0}(1-\delta)}\right)$$

$$= \frac{m^2g}{b^2} + \frac{mV_{x0}}{b} - \frac{mg}{bV_{x0}}\delta - \frac{V_{z0}}{V_{x0}}\delta + \frac{m^2g}{b^2}\ln(1-(1-\delta))$$

$$= \frac{m^2g}{b^2} + \frac{mV_{x0}}{b} + \frac{m^2g}{b^2}\ln\delta \qquad \{\delta \ll 1\}$$

$$-\frac{m^2g}{b^2}\ln\delta = \frac{m^2g}{b^2} + \frac{mV_{x0}}{b}$$

$$\ln\delta = -1 - \frac{bV_{x0}}{mg}$$ V_{x0} is of same order as V_{z0}

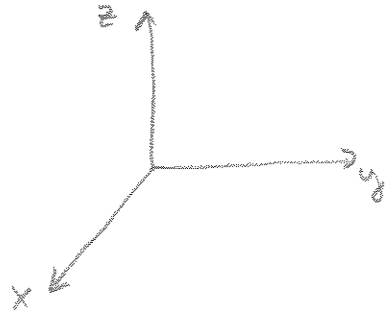
$$\ln\delta \doteq -\frac{bV_{x0}}{mg}$$

$$\ln\delta = \frac{-bV_{x0}}{mg}$$

$$\boxed{\delta = e^{\frac{-bV_{x0}}{mg}}}$$

3-21

$\vec{v}_0 = (v_{x0}, v_{y0}, v_{z0})$ $\quad \vec{v}_w = w\hat{y}$



$$m\frac{d\vec{r}}{dt} = -mg\hat{z} - b\left(\frac{d\vec{r}}{dt} - \vec{v}_w\right)$$

$$= -mg\hat{z} - b\frac{d\vec{r}}{dt} + bw\hat{y}$$

$m\frac{d^2x}{dt^2} = -bv_x$ $\qquad m\frac{d^2y}{dt^2} = -bv_y + bw$ $\qquad m\frac{d^2z}{dt^2} = -mg - bv_z$

(next page)

$$m\frac{dv_x}{dt} = -bv_x$$

$$\int_{v_{x0}}^{v}\frac{dv_x}{v_x} = \int_0^t -\frac{b}{m}dt$$

$$\ln v - \ln v_{x0} = -\frac{b}{m}t$$

$$\ln\left(\frac{v}{v_{x0}}\right) = -\frac{b}{m}t$$

$$\frac{v}{v_{x0}} = e^{-\frac{b}{m}t}$$

$$v = v_{x0}e^{-\frac{b}{m}t}$$

$$\frac{dx}{dt} = v_{x0}e^{-\frac{b}{m}t}$$

$x(0)=0$ $\qquad \int_0^x dx = \int_0^t v_{x0}e^{-\frac{b}{m}t}$

$$\boxed{x = -\frac{v_{x0}m}{b}e^{-\frac{b}{m}t} + \frac{v_{x0}m}{b}}$$

$$\boxed{x = \frac{mv_{x0}}{b}\left(1 - e^{-\frac{b}{m}t}\right)}$$

$$m\frac{dv_y}{dt} = -bv_y + bw$$

$$\frac{dv_y}{dt} = -\frac{b}{m}(v_y - w)$$

$$\int_{v_{y0}}^{v_y}\frac{dv_y}{(v_y - w)} = -\frac{b}{m}t$$

$$\ln\left(\frac{v_y - w}{v_{y0} - w}\right) = -\frac{b}{m}t$$

$$\frac{v_y - w}{v_{y0} - w} = e^{-\frac{b}{m}t}$$

$$v_y - w = (v_{y0} - w)e^{-\frac{b}{m}t}$$

$$v_y = (v_{y0} - w)e^{-\frac{b}{m}t} + w$$

$$\frac{dy}{dt} = (v_{y0} - w)e^{-\frac{b}{m}t} + w$$

$$\int_0^y dy = \int_0^t\left\{(v_{y0} - w)e^{-\frac{b}{m}t} + w\right\}dt$$

$$\boxed{y = -\frac{m}{b}(v_{y0} - w)e^{-\frac{b}{m}t} + wt + \frac{m}{b}(v_{y0} - w)}$$

$$\boxed{y = \frac{m}{b}(v_{y0} - w)\left(1 - e^{-\frac{b}{m}t}\right) + wt}$$

$$m\frac{d^2z}{dt^2} = -mg - bv_z$$

$$m\frac{dv_z}{dt} = -mg - bv_z$$

$$\frac{dv_z}{dt} = -g - \frac{b}{m}v_z$$

$$= -g\left(1 + \frac{b}{mg}v_z\right)$$

$$\frac{dv_z}{\left(1+\frac{b}{mg}v_z\right)} = -g\,dt$$

$$\int_{v_{z0}}^{v_z}\frac{dv_z}{\left(1+\frac{b}{mg}v_z\right)} = -\int_0^t g\,dt$$

$$\frac{mg}{b}\ln\left(1+\frac{b}{mg}v_z\right)\Big|_{v_{z0}}^{v_z} = -gt$$

$$\frac{mg}{b}\ln\left(\frac{mg+bv_z}{mg}\right)\Big|_{v_{z0}}^{v_z} = -gt$$

$$\ln\left(\frac{mg+bv_z}{mg+bv_{z0}}\right) = -\frac{b}{m}t$$

$$\frac{mg+bv_z}{mg+bv_{z0}} = e^{-\frac{b}{m}t}$$

$$mg+bv_z = (mg+bv_{z0})\,e^{-\frac{b}{m}t}$$

$$v_z = \left(\frac{mg}{b}+v_{z0}\right)e^{-\frac{b}{m}t} - \frac{mg}{b}$$

$$\frac{dz}{dt} = \left(\frac{mg}{b}+v_{z0}\right)e^{-\frac{b}{m}t} - \frac{mg}{b}$$

$$\int_0^z dz = \int_0^t\left[\left(\frac{mg}{b}+v_{z0}\right)e^{-\frac{b}{m}t} - \frac{mg}{b}\right]dt$$

$$\boxed{z = -\frac{m}{b}\left(\frac{mg}{b}+v_{z0}\right)e^{-\frac{b}{m}t} - \frac{mg}{b}t + \frac{m}{b}\left(\frac{mg}{b}+v_{z0}\right)}$$

$$\boxed{z = \left(\frac{m^2g}{b^2}+\frac{mv_{z0}}{b}\right)\left(1-e^{-\frac{b}{m}t}\right) - \frac{mg}{b}t}$$

$$x = \frac{V_{x0} m}{b}\left(1 - e^{-\frac{b}{m}t}\right) \qquad y = \frac{m}{b}(V_{y0} - w)\left(1 - e^{-\frac{b}{m}t}\right) + wt$$

$$\frac{bx}{mV_{x0}} = 1 - e^{-\frac{b}{m}t} \qquad \frac{by}{m(V_{y0}-w)} = 1 - e^{-\frac{b}{m}t} + wt$$

$$\frac{bx}{mV_{x0}} - 1 = -e^{-\frac{b}{m}t}$$

$$1 - \frac{bx}{mV_{x0}} = e^{-\frac{b}{m}t}$$

$$\ln\left(1 - \frac{bx}{mV_{x0}}\right) = -\frac{b}{m}t$$

$$-\frac{m}{b}\ln\left(\frac{mV_{x0} - bx}{mV_{x0}}\right) = t$$

$$t = \frac{m}{b}\ln\left(\frac{mV_{x0}}{mV_{x0} - bx}\right)$$

$$z = \left(\frac{m^2 g}{b^2} + \frac{mV_{z0}}{b}\right)\left(1 - e^{-\frac{b}{m}t}\right) - \frac{mg}{b}t$$

$$= \left(\frac{m^2 g}{b^2} + \frac{mV_{z0}}{b}\right)\left(1 - e^{-\frac{b}{m}\left(\frac{m}{b}\ln\left(\frac{mV_{x0}}{mV_{x0}-bx}\right)\right)}\right) - \frac{mg}{b}\left(\frac{m}{b}\ln\left(\frac{mV_{x0}}{mV_{x0}-bx}\right)\right)$$

$$= \left(\frac{m^2 g}{b^2} + \frac{mV_{z0}}{b}\right)\left(1 - \frac{1}{\frac{mV_{x0}}{mV_{x0}-bx}}\right) - \frac{m^2 g}{b^2}\ln\left(\frac{mV_{x0}}{mV_{x0}-bx}\right)$$

$$= \left(\frac{m^2 g}{b^2} + \frac{mV_{z0}}{b}\right)\left(1 - \frac{mV_{x0}-bx}{mV_{x0}}\right) - \frac{m^2 g}{b^2}\ln\left(\frac{mV_{x0}}{mV_{x0}-bx}\right)$$

$$= \left(\frac{m^2 g}{b^2} + \frac{mV_{z0}}{b}\right)\left(\frac{mV_{x0} - mV_{x0} + bx}{mV_{x0}}\right) - \frac{m^2 g}{b^2}\ln\left(\frac{mV_{x0}}{mV_{x0}-bx}\right)$$

$$= \left(\frac{m^2 g}{b^2} + \frac{mV_{z0}}{b}\right)\left(\frac{bx}{mV_{x0}}\right) - \frac{m^2 g}{b^2}\ln\left(\frac{mV_{x0}}{mV_{x0}-bx}\right)$$

$$= \left(\frac{mg}{bV_{x0}} + \frac{V_{z0}}{V_{x0}}\right)x - \frac{m^2 g}{b^2}\ln\left(\frac{mV_{x0}}{mV_{x0}-bx}\right)$$

$$= \left(\frac{mg}{bV_{x0}} + \frac{V_{z0}}{V_{x0}}\right)x + \frac{m^2 g}{b^2}\ln\left(\frac{mV_{x0}-bx}{mV_{x0}}\right)$$

$$z = \left(\frac{mg}{bV_{x0}} + \frac{V_{z0}}{V_{x0}}\right)x + \frac{m^2 g}{b^2}\ln\left(1 - \frac{bx}{mV_{x0}}\right)$$

$$\frac{bx}{mV_{x0}} \ll 1$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

$$z = \left(\frac{mg}{bV_{x0}} + \frac{V_{z0}}{V_{x0}}\right)x + \frac{m^2g}{b^2}\left[\frac{-bx}{mV_{x0}} - \frac{b^2}{2m^2V_{x0}^2}x^2 - \frac{b^3}{3m^3V_{x0}^3}x^3 - \cdots\right]$$

$$= \frac{mg}{bV_{x0}}x + \frac{V_{z0}}{V_{x0}}x - \frac{mg}{bV_{x0}}x - \frac{g}{2V_{x0}^2}x^2 - \frac{bg}{3mV_{x0}^3}x^3 -$$

$$= \frac{V_{z0}}{V_{x0}}x - \frac{g}{2V_{x0}^2}x^2 - \frac{bg}{3mV_{x0}^3}x^3$$

$$= x\left(\frac{V_{z0}}{V_{x0}} - \frac{g}{2V_{x0}^2}x - \frac{bg}{3mV_{x0}^3}x^2\right)$$

Setting $z=0$ gives x_m

$$0 = \frac{V_{z0}}{V_{x0}} - \frac{g}{2V_{x0}^2}x - \frac{bg}{3mV_{x0}^3}x^2$$

$$\frac{g}{2V_{x0}^2}x_m = \frac{V_{z0}}{V_{x0}} - \frac{bg}{3mV_{x0}^3}x_m^2$$

$$x_m = \frac{2V_{z0}V_{x0}}{g} - \frac{2}{3}\frac{b}{mV_{x0}}x_m^2$$

First approximation

$$x_m \doteq \frac{2V_{z0}V_{x0}}{g}$$

Substitute this approximation into the previous equation to find first order terms of b.

$$x_m \doteq \frac{2V_{z0}V_{x0}}{g} - \frac{2}{3}\frac{b}{mV_{x0}}\left(\frac{2V_{z0}V_{x0}}{g}\right)^2$$

$$x_m \doteq \frac{2V_{z0}V_{x0}}{g} - \frac{8}{3}\frac{bV_{z0}^2V_{x0}}{mg^2}$$

$$\boxed{x_{1m} = \frac{2V_{z0}V_{x0}}{g} - \frac{8}{3}\frac{bV_{z0}^2V_{x0}}{mg^2}}$$

$$t_m = \frac{m}{b}\ln\left(\frac{mV_{x0}}{mV_{x0}-bx_m}\right)$$

$$y_m = \frac{m}{b}(V_{y0}-w)\left(1-e^{-\frac{b}{m}t}\right) + wt_m$$

$$= \frac{m}{b}(V_{y0}-w)\left(1 - \frac{1}{\frac{mV_{x0}}{mV_{x0}-bx_m}}\right) + w\frac{m}{b}\ln\left(\frac{mV_{x0}}{mV_{x0}-bx_m}\right)$$

$$= \frac{m}{b}(V_{y0}-w)\left(1 - \frac{mV_{x0}-bx_m}{mV_{x0}}\right) - w\frac{m}{b}\ln\left(\frac{mV_{x0}-bx_m}{mV_{x0}}\right)$$

$$= \frac{m}{b}(V_{y0}-w)\left(\frac{mV_{x0}-mV_{x0}+bx_m}{mV_{x0}}\right) - w\frac{m}{b}\ln\left(1-\frac{bx_m}{mV_{x0}}\right)$$

$$= \frac{m}{b}(V_{y0}-w)\left(\frac{bx_m}{mV_{x0}}\right) - w\frac{m}{b}\ln\left(1-\frac{bx_m}{mV_{x0}}\right)$$

$$y_m = \frac{V_{y0}-w}{V_{x0}}x_m - w\frac{m}{b}\ln\left(1-\frac{bx_m}{mV_{x0}}\right) \qquad \frac{bx}{mV_{x0}} \ll 1$$

$$-\frac{wm}{b}\ln\left(1-\frac{bx_m}{mV_{x0}}\right) = -w\frac{m}{b}\left(\frac{-bx_m}{mV_{x0}} - \frac{b^2}{2m^2V_{x0}^2}x_m^2 - \frac{b^3}{3m^3V_{x0}^3}x_m^3 - \cdots\right)$$

$$= -w\left(\frac{-x_m}{V_{x0}} - \frac{b}{2mV_{x0}^2}x_m^2 - \text{higher orders of } b\right)$$

$$= -w\left(\left(\frac{-2V_{z0}}{g} + \frac{8}{3}\frac{bV_{z0}^2}{mg^2}\right) - \frac{b}{2mV_{x0}^2}\left[\frac{2V_{z0}V_{x0}}{g} - \frac{8}{3}\frac{bV_{z0}^2V_{x0}}{mg^2}\right]^2\right)$$

$$= -w\left(\frac{-2V_{z0}}{g} + \frac{8}{3}\frac{bV_{z0}^2}{mg^2} - \frac{b}{2mV_{x0}^2}\left(\frac{4V_{z0}^2V_{x0}^2}{g^2} + b \text{ and higher terms}\right)\right)$$

$$= -w\left(\frac{-2V_{z0}}{g} + \frac{8}{3}\frac{bV_{z0}^2}{mg^2} - \frac{2bV_{z0}^2}{mg^2}\right)$$

$$y_m = \frac{V_{y0}-w}{V_{x0}}\left(\frac{2V_{z0}V_{x0}}{g} - \frac{8}{3}\frac{bV_{z0}^2V_{x0}}{mg^2}\right) - w\left(\frac{-2V_{z0}}{g} + \frac{8}{3}\frac{bV_{z0}^2}{mg^2} - \frac{2bV_{z0}^2}{mg^2}\right)$$

$$= (V_{y0}-w)\left(\frac{2V_{z0}}{g} - \frac{8}{3}\frac{bV_{z0}^2}{mg^2}\right) + w\left(\frac{2V_{z0}}{g} - \frac{8}{3}\frac{bV_{z0}^2}{mg^2} + \frac{2bV_{z0}^2}{mg^2}\right)$$

$$= \frac{2V_{z0}V_{y0}}{g} - \frac{8}{3}\frac{bV_{z0}^2V_{y0}}{mg^2} - \frac{2V_{z0}w}{g} + \frac{8}{3}\frac{bV_{z0}^2w}{mg^2} + \frac{2V_{z0}w}{g} - \frac{8}{3}\frac{bV_{z0}^2w}{mg^2} + \frac{2bV_{z0}^2w}{mg^2}$$

$$y_m = \frac{2V_{z0}V_{y0}}{g} - \frac{8}{3}\frac{bV_{z0}^2V_{y0}}{mg^2} + \frac{2bV_{z0}^2w}{mg^2}$$

$$\boxed{y_m = \frac{2V_{z0}V_{y0}}{g} - \frac{8}{3}\frac{bV_{z0}^2V_{y0}}{mg^2} + \frac{2bV_{z0}^2w}{mg^2}}$$

Neglecting b and $\vec{V}_w$

$$x = V_{x0}t$$

$$y = V_{y0}t$$

$$z = V_{z0}t - \frac{1}{2}gt^2$$

$$z_{max} = \frac{V_{z0}^2}{2g} \rightarrow -\frac{1}{2}gt_m^2 + V_{z0}t_m - \frac{V_{z0}^2}{2g} = 0$$

$$t_m^2 - \frac{2V_{z0}}{g}t_m + \frac{V_{z0}^2}{g^2} = 0$$

$$\left(t_m - \frac{V_{z0}}{g}\right)^2 = 0$$

$$t_m = \frac{V_{z0}}{g}$$

$$t_r = 2t_m$$

$$t_r = \frac{2V_{z0}}{g}$$

$$x_m = \frac{2V_{z0}V_{x0}}{g} \qquad y_m = \frac{2V_{z0}V_{y0}}{g}$$

with air resistance alone

$$x_{1m_b} = \frac{2V_{z0}V_{x0}}{g} - \frac{8}{3}\frac{bV_{z0}^2V_{x0}}{mg^2}$$

$$\frac{x_{1m_b} - x_m}{x_m} = \frac{-\frac{8}{3}\frac{bV_{z0}^2V_{x0}}{mg^2}}{\frac{2V_{z0}V_{x0}}{g}} = -\frac{8}{3}\frac{bV_{z0}^2V_{x0}}{mg^2} \cdot \frac{g}{2V_{z0}V_{x0}}$$

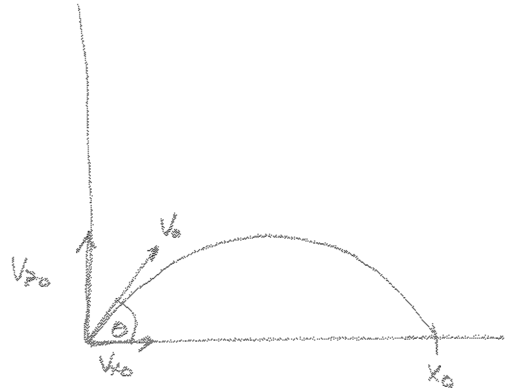
$$\boxed{\frac{x_{1m_b} - x_m}{x_m} = \frac{-4bV_{z0}}{3mg}}$$

$$y_{1m_b} = \frac{2V_{z0}V_{y0}}{g^2} - \frac{8}{3}\frac{bV_{z0}V_{y0}}{mg^2}, \quad w = 0$$

$$\boxed{\frac{y_{1m_b} - y_m}{y_m} = \frac{-4bV_{z0}}{3mg}}$$

$$\boxed{w \neq 0 \text{ additional miss } = \frac{2bV_{z0}^2 w}{mg^2}}$$

3-33



$v_{z0} = v_0 \sin\theta \quad v_{x0} = v_0 \cos\theta$

a) $z = -\frac{1}{2} g t^2 + v_{z0} t$

$= t\left(-\frac{1}{2} g t + v_{z0}\right)$

$0 = t\left(-\frac{1}{2} g t + v_{z0}\right)$

$t = 0$ or

$t = \frac{2 v_{z0}}{g}$

$= \frac{2 v_0 \sin\theta}{g}$

$x = v_{x0} t$

$= (v_0 \cos\theta)\left(\frac{2 v_0 \sin\theta}{g}\right)$

$= \frac{v_0^2 \, 2 \sin\theta \cos\theta}{g}$

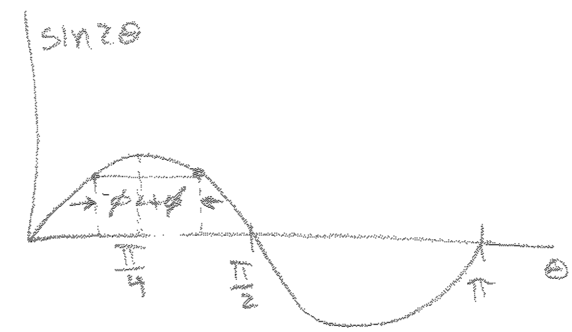
$x = \frac{v_0^2 \sin 2\theta}{g}$

$x = x_m$ at $\theta_m = \frac{\pi}{4}$

$$\boxed{\theta = \sin^{-1}\left(\frac{xg}{v_0^2}\right) \Big/ 2}$$

$\frac{xg}{v_0^2} = 2\cos^2\phi - 1, \quad x < x_m \quad \theta = \frac{\pi}{4} \pm \phi$

$$\boxed{\pm\left(\frac{xg}{2 v_0^2} + \frac{1}{2}\right)^{1/2} = \cos\phi, \quad \cos\phi = \cos(-\phi)}$$



$\sin 2\theta = \sin(2(\theta_m \pm \phi))$

$= \sin\left(\frac{\pi}{2} \pm 2\phi\right)$

$= \sin\frac{\pi}{2} \cos 2\phi \pm \sin 2\phi \cos\frac{\pi}{2}$

$= \cos 2\phi$

$= 2\cos^2\phi - 1$

b) First order correction

$$x \doteq \frac{2V_{x_0}V_{z_0}}{g} - \frac{8}{3}\frac{bV_{z_0}^2 V_{x_0}}{mg^2}$$

$$\Delta x = -\frac{8}{3}\frac{bV_{z_0}^2 V_{x_0}}{mg^2}$$

with $b=0$

$$x = \frac{2V_0^2 \sin\theta\cos\theta}{g}$$

$$\frac{dx}{d\theta} = \frac{2V_0^2}{g}(-\sin^2\theta + \cos^2\theta)$$

$$\Delta x = \frac{2V_0^2}{g}(\cos^2\theta - \sin^2\theta)\,\Delta\theta$$

To offset Δx_1 $-\Delta x$ must be found

$$\frac{8}{3}\frac{b}{mg^2}V_0^2\sin^2\theta\, V_0\cos\theta = \frac{2V_0^2}{g}(\cos^2\theta - \sin^2\theta)\,\Delta\theta$$

$$\frac{4}{3}\frac{bV_0}{mg}\cos\theta = \frac{(\cos^2\theta - \sin^2\theta)}{\sin^2\theta}\Delta\theta$$

$$= (\cot^2\theta - 1)\,\Delta\theta$$

$$\boxed{\Delta\theta = \frac{4bV_0\cos\theta}{3mg(\cot^2\theta - 1)}}$$

3-35

a) $F_x = 6abz^3y - 20bx^3y^2$, $F_y = 6abxz^3 - 10bx^4y$, $F_z = 18abxz^2y$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 6abz^3y - 20bx^3y^2 & 6abxz^3 - 10bx^4y & 18abxz^2y \end{vmatrix}$$

$$= \hat{x}\left(18abxz^2 - (18abxz^2)\right) + \hat{y}\left(18abz^2y - 18abz^2y\right) + \hat{z}\left(6abz^3 - 40bx^3y - (6abz^3 - 40bx^3y)\right)$$

$= 0 \rightarrow$ conservative force

$$V = -\int_{r_s}^{r} \vec{F} \cdot d\vec{r} \qquad d\vec{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}$$

$$= -\int_{x=0,y=0,z=0}^{x=x_0,y=0,z=0} (6abz^3y - 20bx^3y^2)\,dx - \int_{x=x_0,y=0,z=0}^{x=x_0,y=y_0,z=0} (6abxz^3 - 10bx^4y)\,dy - \int_{x=x_0,y=y_0,z=0}^{x=x_0,y=y_0,z=z_0} 18abxz^2y\,dz$$

$$= 0 - (-5x_0^4y_0^2) - 6abx_0y_0z_0^3$$

$$\boxed{V = 5x^4y^2 - 6abxyz^3}$$

b) $F_x = 18abyz^3 - 20bx^3y^2$, $F_y = 18abxz^3 - 10bx^4y$, $F_z = 6abxyz^2$

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 18abyz^3 - 20bx^3y^2 & 18abxz^3 - 10bx^4y & 6abxyz^2 \end{vmatrix}$$

$$= \hat{x}\left(6abxz^2 - 56abxz^2\right) + $$

$\neq 0 \rightarrow$ not conservative

c) $\vec{F} = F_x(x)\,\hat{x} + F_y(y)\,\hat{y} + F_z(z)\,\hat{z}$

Since $\vec{F}$ only has components in the unit vector directions

$\vec{\nabla} \times \vec{F} = 0$

$$\boxed{V = -\int_{x_s}^{x} F_x(x)\,dx - \int_{y_s}^{y} F_y(y)\,dy - \int_{z_s}^{z} F_z(z)\,dz}$$

3-37

a) $F_x = axe^{-R}$, $F_y = bye^{-R}$, $F_z = cze^{-R}$, $R = ax^2+by^2+cz^2$

$$V = -\int_{r_0}^{r} \vec{F}\cdot d\vec{r} \qquad d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$= -\int_{x=0,y=0,z=0}^{x=x_0,y=0,z=0} axe^{-ax^2}dx - \int_{x=x_0,y=0,z=0}^{x=x_0,y=y_0,z=0} bye^{-ax_0^2-by^2}dy - \int_{x=x_0,y=y_0,z=0}^{x=x_0,y=y_0,z=z_0} cze^{-ax_0^2-by_0^2-cz^2}dz$$

$u = -ax^2$ $\quad$ $u = -ax_0^2-by^2$ $\quad$ $u = -ax_0^2-by_0^2-cz^2$

$du = -2ax\,dx$ $\quad$ $du = -2by\,dy$ $\quad$ $du = -2cz\,dz$

$-\frac{du}{2} = ax\,dx$ $\quad$ $-\frac{du}{2} = by\,dy$ $\quad$ $-\frac{du}{2} = cz\,dz$

$$V = \frac{e^{-ax^2}}{2}\Big|_0^{x_0} + \frac{e^{-ax_0^2-by^2}}{2}\Big|_0^{y_0} + \frac{e^{-ax_0^2-by_0^2-cz^2}}{2}\Big|_0^{z_0}$$

$$= \frac{1}{2}\left[e^{-ax_0}-1 + e^{-ax_0-by_0^2} - e^{-ax_0} + e^{-ax_0^2-by_0^2-cz_0^2} - e^{-ax_0^2-by_0^2}\right]$$

$$= \frac{1}{2}\left[e^{-ax^2-by^2-cz^2}-1\right]$$

$= \frac{1}{2}e^{-R} - \frac{1}{2}$, $V=0$ at $x=y=z=0$, so the constant can be dropped since $F = -\vec{\nabla}V$

$$\boxed{V = \frac{1}{2}e^{-R}}$$

b) $\vec{F} = \vec{A}f(\vec{A}\cdot\vec{r})$ where $\vec{A}$ is a constant vector, $f(s)$ where $s = \vec{A}\cdot\vec{r}$

$$V = -\int_{r_0}^{r} \vec{F}\cdot d\vec{r}$$

$= -\int f(s)\vec{A}\cdot d\vec{r}$ $\quad$ $s = \vec{A}\cdot r$, $ds = \vec{A}\cdot d\vec{r}$

$$\boxed{V = -\int_0^s f(s)\,ds}$$

c) $\vec{F} = (\vec{r} \times \vec{A}) f(\vec{A} \cdot \vec{r})$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$

$$\vec{A} = A_x\hat{x} + A_y\hat{y} + A_z\hat{z}$$

$$(\vec{r} \times \vec{A}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ A_x & A_y & A_z \end{vmatrix}$$

$$= (yA_z - zA_y)\hat{x} + (zA_x - xA_z)\hat{y} + (xA_y - yA_x)\hat{z}$$

$$\vec{A} \cdot \vec{r} = xA_x + yA_y + zA_z$$

$$F_x = (yA_z - zA_y) f(xA_x + yA_y + zA_z)$$

$$F_y = (zA_x - xA_z) f(xA_x + yA_y + zA_z)$$

$$F_z = (xA_y - yA_x) f(xA_x + yA_y + zA_z)$$

$$V = -\int_{r_s}^{r} \vec{F} \cdot d\vec{r} \qquad d\vec{r} = dx\hat{x} + dy\hat{y} + dz\hat{z}$$

$$= -\int_{x=0,y=0,z=0}^{x=x_0,y=0,z=0} (yA_z - zA_y) f(\vec{A}\cdot\vec{r})\,dx - \int_{x=x_0,y=0,z=0}^{x=x_0,y=y_0,z=0} (zA_x - xA_z) f(\vec{A}\cdot\vec{r})\,dy - \int_{x=x_0,y=y_0,z=0}^{x=x_0,y=y_0,z=z_0} (xA_y - yA_x) f(\vec{A}\cdot\vec{r})\,dz$$

$$= 0 + x_0A_z \int f(x_0A_x + yA_y)\,dy - (x_0A_y - y_0A_x) \int f(x_0A_x + y_0A_y + zA_z)\,dz$$

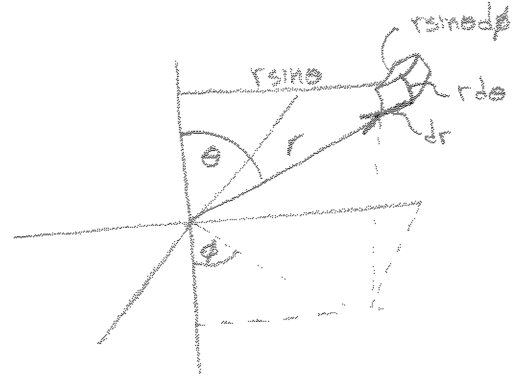
$$\boxed{V = xA_z \int_0^{y} f(xA_x + yA_y)\,dy - (xA_y - yA_x) \int_0^{z} f(xA_x + yA_y + zA_z)\,dz}$$

3-39

$$\vec{F} = \hat{r}\,F(r)$$

$$d\vec{r} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\theta}$$

$$\vec{F} = F(r)\,\hat{r} + 0\,\hat{\theta} + 0\,\hat{\phi}$$

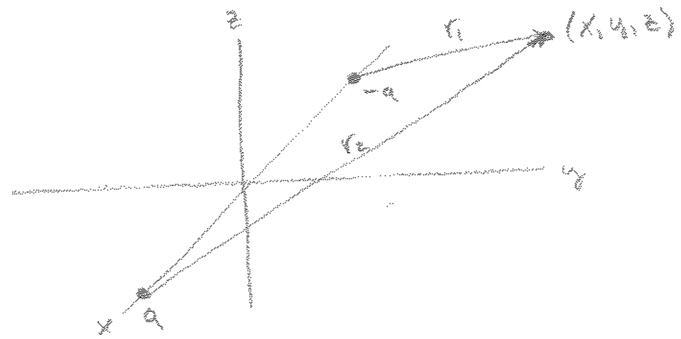


$$\int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} = \int_{r_1}^{r_2} F_r\,dr$$

$$= V(r_2) - V(r_1) \quad \left\{\text{Fundamental Theorem of Calculus}\right\}$$

3-41

$$V = -\frac{e^2}{r_1} - \frac{e^2}{r_2}$$



$$\vec{r}_1 = (x+a)\hat{x} + y\hat{y} + z\hat{z} \qquad \vec{r}_2 = (x-a)\hat{x} + y\hat{y} + z\hat{z}$$

$$F = -\vec{\nabla} V \qquad r_1 = \left((x+a)^2 + y^2 + z^2\right)^{1/2}$$

$$= -\frac{\partial V}{\partial x}\hat{x} - \frac{\partial V}{\partial y}\hat{y} - \frac{\partial V}{\partial z}\hat{z} \qquad r_2 = \left((x-a)^2 + y^2 + z^2\right)^{1/2}$$

$$F_x = -\frac{\partial V}{\partial x}$$

$$= -\frac{\partial}{\partial x}\left(\frac{-e^2}{r_1} - \frac{e^2}{r_2}\right)$$

$$= e^2\frac{\partial}{\partial x}\left(\frac{1}{r_1} + \frac{1}{r_2}\right)$$

$$= e^2\frac{\partial}{\partial x}\left(\frac{1}{((x+a)^2+y^2+z^2)^{1/2}} + \frac{1}{((x-a)^2+y^2+z^2)^{1/2}}\right)$$

$$= e^2\frac{\partial}{\partial x}\left(((x+a)^2+y^2+z^2)^{-1/2} + ((x-a)^2+y^2+z^2)^{-1/2}\right)$$

$$= e^2\left[-\frac{1}{2}((x+a)^2+y^2+z^2)^{-3/2} 2(x+a) - \frac{1}{2}((x-a)^2+y^2+z^2)^{-3/2} 2(x-a)\right]$$

$$= e^2\left[-(x+a)r_1^{-3} - (x-a)r_2^{-3}\right]$$

$$= e^2\left[-xr_1^{-3} - ar_1^{-3} - xr_2^{-3} + ar_2^{-3}\right]$$

$$\boxed{F_x = e^2\left[-a(r_1^{-3} - r_2^{-3}) - x(r_1^{-3} + r_2^{-3})\right]}$$

$$F_y = -\frac{\partial V}{\partial y}$$

$$= e^2 \frac{\partial}{\partial y}\left(\left((x+a)^2+y^2+z^2\right)^{-1/2} + \left((x-a)^2+y^2+z^2\right)^{-1/2}\right)$$

$$= e^2\left[-\frac{1}{2} r_1^{-3} 2y - \frac{1}{2} r_2^{-3} 2y\right]$$

$$= -y e^2\left[r_1^{-3} + r_2^{-3}\right]$$

$$F_z = -\frac{\partial V}{\partial z}$$

$$= e^2 \frac{\partial}{\partial z}\left(\left((x+a)^2+y^2+z^2\right)^{-1/2} + \left((x-a)^2+y^2+z^2\right)^{-1/2}\right)$$

$$= -z e^2\left[r_1^{-3} + r_2^{-3}\right]$$

3-43 $\quad V(r) = \frac{1}{2}kr^2$

$$'V'(r) = V(r) + \frac{L^2}{2mr^2}$$

$$= \frac{1}{2}kr^2 + \frac{L^2}{2mr^2}$$

$$\frac{d'V'}{dr} = kr - \frac{L^2}{m}\frac{1}{r^3}$$

$$0 = \frac{kr^4 - \frac{L^2}{m}}{r^3}$$

$$kr_0^4 = \frac{L^2}{m}$$

$$r_0^4 = \frac{L^2}{km}$$

$$r_0^2 = \pm \frac{L}{(km)^{1/2}}$$

$$r_0 = \pm \frac{L^{1/2}}{(km)^{1/4}}$$

$$'V'\left(\frac{L^{1/2}}{(km)^{1/4}}\right) = \frac{1}{2}L\left(\frac{k}{m}\right)^{1/2} + \frac{1}{2}L\left(\frac{k}{m}\right)^{1/2}$$

$$= L\left(\frac{k}{m}\right)^{1/2}$$

$$'V'\left(-\frac{L^{1/2}}{(km)^{1/4}}\right) = L\left(\frac{k}{m}\right)^{1/2}$$

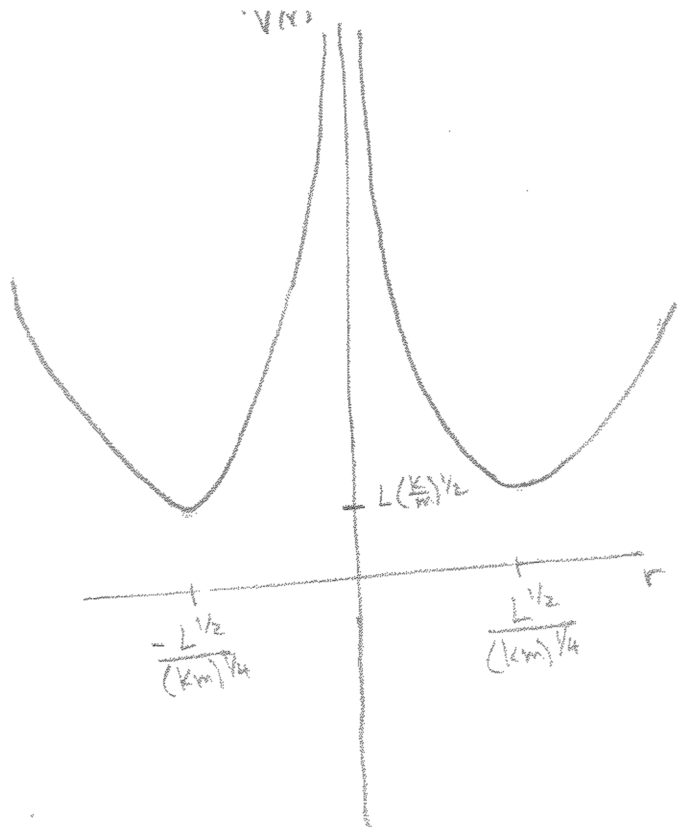
For $E = L\left(\frac{k}{m}\right)^{1/2}$, $r_0^2 = \frac{L}{(km)^{1/2}}$

$$\dot{\theta} = \frac{L}{mr_0^2}$$

$$= \frac{L}{m}\frac{1}{\frac{L}{(km)^{1/2}}}$$

$$= \frac{(km)^{1/2}}{m}$$

$$\boxed{\dot{\theta} = \left(\frac{k}{m}\right)^{1/2}}$$



$$\omega^2 = \frac{1}{m}\left(\frac{d^2\,'V'}{dr^2}\right)_{r_0}, \quad E \gtrsim L\left(\frac{k}{m}\right)^{1/2}$$

$$\frac{d^2\,'V'}{dr^2} = \frac{d}{dr}\left(kr - \frac{L^2}{m}\frac{1}{r^3}\right)$$

$$= k + \frac{3L^2}{m}\frac{1}{r^4}$$

$$\left(\frac{d^2\,'V'}{dr^2}\right)_{r_0} = k + \frac{3L^2}{m}\frac{1}{\frac{L^2}{km}}$$

$$= k + 3k$$

$$\omega^2 = \frac{4k}{m}$$

$$\boxed{\omega_r = 2\sqrt{\frac{k}{m}}}$$

Nature of solutions:
For small E orbit is nearly circular.
For large E, $\dot{r}_{r<r_0} > \dot{r}_{r>r_0}$ giving an ellipse.
The orbits for $E \geq L\left(\frac{k}{m}\right)^{1/2}$ are bounded with two turning points.

3-45 $\qquad V(r) = \frac{1}{2}kr^2$

$$\int_{r_0}^{r} \frac{dr}{\left[E - V(r) - \frac{L^2}{2mr^2}\right]^{1/2}} = \sqrt{\frac{2}{m}}\,t$$

$$E - \frac{1}{2}kr^2 - \frac{L^2}{2mr^2} = \frac{2Emr^2 - kmr^4 - L^2}{2mr^2} = \frac{-kmr^4 + 2Emr^2 - L^2}{2mr^2}$$

$$\int_{r_0}^{r} \frac{dr}{\left[E - \frac{1}{2}kr^2 - \frac{L^2}{2mr^2}\right]^{1/2}} = \sqrt{2m}\int_{r_0}^{r} \frac{r\,dr}{\left(-kmr^4 + 2Emr^2 - L^2\right)^{1/2}} =$$

$$-kmr^4 + 2Emr^2 - L^2 =$$

$$-km\left(r^4 - \frac{2Er^2}{k} + \frac{L^2}{km}\right) =$$

$$-km\left[r^4 - \frac{2Er^2}{k} + \frac{E^2}{k^2} + \frac{L^2}{km} - \frac{E^2}{k^2}\right] =$$

$$-km\left[\left(r^2 - \frac{E}{k}\right)^2 - \left(\frac{E^2}{k^2} - \frac{L^2}{km}\right)\right] =$$

$$km\left[\frac{E^2}{k^2} - \frac{L^2}{km} - \left(r^2 - \frac{E}{k}\right)^2\right]$$

$$\sqrt{2m}\int_{r_0}^{r} \frac{r\,dr}{\left[km\left(\frac{E^2}{k^2} - \frac{L^2}{km} - \left(r^2 - \frac{E}{k}\right)^2\right)\right]^{1/2}} =$$

$$\sqrt{\frac{2}{k}}\int_{r_0}^{r} \frac{r\,dr}{\left[\frac{E^2}{k^2} - \frac{L^2}{km} - \left(r^2 - \frac{E}{k}\right)^2\right]^{1/2}} =$$

let $r^2 - \frac{E}{k} = \frac{\sqrt{E^2 - L^2\frac{k}{m}}}{k}\sin\theta$

$$2r\,dr = \frac{\sqrt{E^2 - L^2\frac{k}{m}}}{k}\cos\theta\,d\theta$$

$$\frac{1}{2}\sqrt{\frac{2}{k}}\int_{\theta_0}^{\theta} \frac{\frac{\sqrt{E^2 - L^2\frac{k}{m}}}{k}\cos\theta\,d\theta}{\frac{\sqrt{E^2 - L^2\frac{k}{m}}}{k}\left(1 - \sin^2\theta\right)^{1/2}}$$

$$\sqrt{\frac{1}{2k}}\int_{\theta_0}^{\theta} d\theta = \sqrt{\frac{1}{2k}}(\theta - \theta_0)$$

$$\sqrt{\frac{1}{2k}}\,(\theta - \theta_0) = \sqrt{\frac{2}{m}}\,t$$

$$\frac{1}{2}(\theta - \theta_0) = \sqrt{\frac{k}{m}}\,t$$

$$\theta = 2\omega t + \theta_0$$

$$r^2 = \frac{E}{k} - \frac{\left(E^2 - L^2\frac{k}{m}\right)^{1/2}}{k}\sin\theta$$

$$\boxed{kr^2 = E - \left(E^2 - L^2\omega^2\right)^{1/2}\sin(2\omega t + \theta_0), \quad \omega = \sqrt{\frac{k}{m}}}$$

3-47

$$V(r) = \frac{Ke^{-\alpha r}}{r} \qquad K<0$$

a) $F = -\vec{\nabla} V$

$$= -\frac{\partial V}{\partial r}\hat{r}$$

$$= -\frac{\partial}{\partial r}\left(\frac{Ke^{-\alpha r}}{r}\right)\hat{r}$$

$$= -\frac{\left(r(-\alpha Ke^{-\alpha r}) - Ke^{-\alpha r}\right)}{r^2}\hat{r}$$

$$= \frac{r\alpha Ke^{-\alpha r} + Ke^{-\alpha r}}{r^2}\hat{r}$$

$$\boxed{\vec{F} = \frac{(1+\alpha r)Ke^{-\alpha r}}{r^2}\hat{r}}$$

The inverse square law force, $\vec{F}_{inv} = \frac{K}{r^2}\hat{r}$, and the above force are related by

$$\vec{F} = \vec{F}_{inv}(1+\alpha r)e^{-\alpha r}$$

So $\vec{F}$ decreases by a factor of $(1+\alpha r)e^{-\alpha r}$ compared to $\vec{F}_{inv}$ as r increases.

b) $V(r) = \frac{Ke^{-\alpha r}}{r}$

$$\frac{dV(r)}{dr} = \frac{-(1+\alpha r)Ke^{-\alpha r}}{r^2}$$

$$0 = (1+\alpha r)Ke^{\alpha r}$$

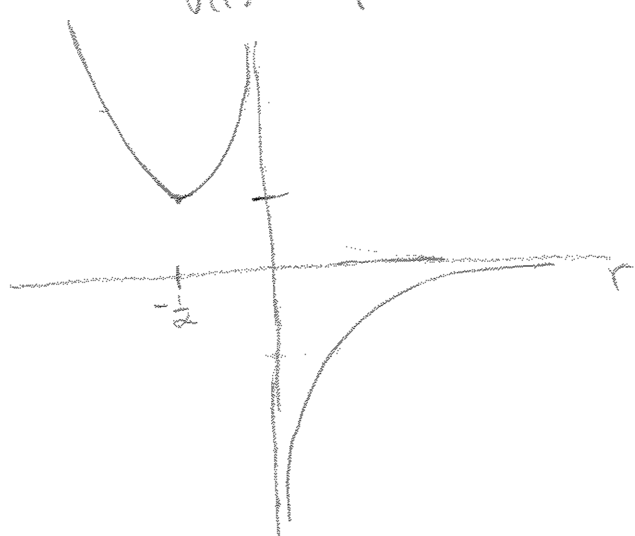
$$0 = (1+\alpha r)$$

$$r_0 = -\frac{1}{\alpha}$$

$$V(r_0) = \frac{Ke^{1}}{-\frac{1}{\alpha}}$$

$$= \alpha Ke^{1},\ K<0$$

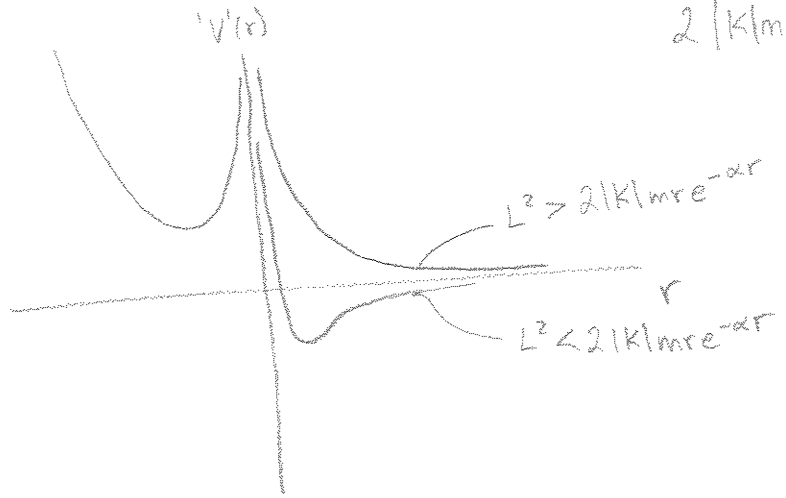
$$V(r) = \frac{Ke^{-\alpha r}}{r}$$



$$V'(r) = \frac{Ke^{-\alpha r}}{r} + \frac{L^2}{2mr^2}$$

$$0 > \frac{Ke^{-\alpha r}}{r} + \frac{L^2}{2mr^2}$$

$$2|K|mre^{-\alpha r} > L^2$$



$E > 0$, one turning pt., open orbit $r \to \infty$

$E < 0$, two turning pts., closed orbit

d) $$\frac{dV'}{dr} = \frac{-(1+\alpha r)Ke^{-\alpha r}}{r^2} - \frac{L^2}{mr^3}$$

For $r = a$, $\frac{dV'}{dr} = 0$ to give a circular orbit

$$0 = \frac{-(1+\alpha a)Ke^{-\alpha a}}{a^2} - \frac{L^2}{ma^3}$$

$$\boxed{L^2 = -(1+\alpha a)Ke^{-\alpha a}ma}$$

$$E = \frac{Ke^{-\alpha r}}{r} + \frac{L^2}{2mr^2} \quad \text{at } r = a$$

$$E = \frac{Ke^{-\alpha a}}{a} + \frac{-maK(1+\alpha a)e^{-\alpha a}}{2ma^2}$$

$$= \frac{Ke^{-\alpha a}}{a} - \frac{K(1+\alpha a)e^{-\alpha a}}{2a}$$

$$= \frac{2Ke^{-\alpha a} - Ke^{-\alpha a} - K\alpha ae^{-\alpha a}}{2a}$$

$$= \frac{Ke^{-\alpha a} - K\alpha ae^{-\alpha a}}{2a}$$

$$\boxed{E = \frac{Ke^{-\alpha a}(1-\alpha a)}{2a}}$$

e) i) $S = \frac{L\tau}{2m}$ $\quad S = \pi a^2$

$$\tau = \frac{2mS}{L}$$

$$= \frac{2\pi a^2 m}{\left(-(1+\alpha a)e^{-\alpha a}Kma\right)^{1/2}}$$

$$\boxed{\tau_c = 2\pi\left[-K(1+\alpha a)e^{-\alpha a}/ma^3\right]^{-\frac{1}{2}}}$$

ii)

$$\omega^2 = \frac{1}{m}\left(\frac{d^2V}{dr^2}\right)_{r=a}$$

$$\frac{d^2V}{dr^2} = \frac{d}{dr}\left(\frac{-(1+\alpha r)Ke^{-\alpha r}}{r^2} - \frac{L^2}{mr^3}\right)$$

$$= \frac{d}{dr}\left(\frac{-Ke^{-\alpha r}}{r^2} - \frac{K\alpha re^{-\alpha r}}{r^2} - \frac{L^2}{mr^2}\right)$$

$$= \frac{r^2\alpha Ke^{-\alpha r} + 2rKe^{-\alpha r}}{r^4} - \frac{K\alpha r^2(-\alpha re^{-\alpha r} + e^{-\alpha r}) - K\alpha re^{-\alpha r}2r}{r^4} + \frac{3L^2}{mr^4}$$

$$= \frac{K\alpha r^2e^{-\alpha r} + 2Kre^{-\alpha r}}{r^4} - \frac{K\alpha r^2(-\alpha re^{-\alpha r} + e^{-\alpha r})}{r^4} + \frac{2K\alpha r^2e^{-\alpha r}}{r^4} + \frac{3L^2}{mr^4}$$

$$= \frac{K\alpha r^2e^{-\alpha r} + 2Kre^{-\alpha r} + K\alpha^2r^3e^{-\alpha r} - K\alpha r^2e^{-\alpha r} - 2K\alpha r^2e^{-\alpha r}}{r^4} + \frac{3L^2}{mr^4}$$

$$= \frac{2Kre^{-\alpha r} + 2K\alpha r^2e^{-\alpha r} + K\alpha^2r^3e^{-\alpha r}}{r^4} + \frac{3L^2}{mr^4}$$

$$\omega^2 = \frac{1}{m}\left[\frac{2Kae^{-\alpha a} + 2K\alpha a^2e^{-\alpha a} + K\alpha^2a^3e^{-\alpha a}}{a^4} + \frac{3\left(-(1+\alpha a)e^{-\alpha a}Kma\right)}{ma^4}\right]$$

$$= \frac{2Kae^{-\alpha a} + 2K\alpha a^2e^{-\alpha a} + K\alpha^2a^3e^{-\alpha a} - 3Kae^{-\alpha a} - 3K\alpha a^2e^{-\alpha a}}{ma^4}$$

$$= \frac{-Kae^{-\alpha a} - K\alpha a^2e^{-\alpha a} + K\alpha^2a^3e^{-\alpha a}}{ma^4}$$

$$\omega^2 = \frac{-Ka(1+\alpha a-\alpha^2a^2)e^{-\alpha a}}{ma^4}$$

$$= \frac{-K(1+\alpha a-\alpha^2a^2)e^{-\alpha a}}{ma^3}$$

$$\frac{2\pi}{\tau} = \omega = \left(\frac{-K(1+\alpha a-\alpha^2a^2)e^{-\alpha a}}{ma^3}\right)^{1/2}$$

$$\boxed{\tau_r = 2\pi\left(\frac{-K(1+\alpha a-\alpha^2a^2)e^{-\alpha a}}{ma^3}\right)^{-\frac{1}{2}}}$$

f) For closed orbits

$$\omega_c = \omega_r \rightarrow \tau_c = \tau_r$$

when a is small, in order for τ_c to nearly equal τ_r

$$(1+\alpha a) \cong (1+\alpha a-\alpha^2a^2)$$

So,

$$\alpha^2a^2 < 1+\alpha a$$

$$(\alpha a)^2-\alpha a-1<0$$

$$(\alpha a)^2-\alpha a+\tfrac{1}{4}<1+\tfrac{1}{4}$$

$$(\alpha a-\tfrac{1}{2})^2<\tfrac{5}{4}$$

$$\alpha a-\tfrac{1}{2}<\tfrac{\sqrt{5}}{2}$$

$$\boxed{\alpha a < \frac{1+\sqrt{5}}{2} \text{ for stable orbits}}$$

3-49

$$F' = -mkr \qquad k = -\frac{4}{3}\pi\rho G$$

$$F = \frac{k_1}{r^2} \qquad k_1 = -GMm$$

$$V(r) = -\int \frac{k_1}{r^2}\,dr \qquad V'(r) = -\int -\frac{4}{3}\pi\rho G m r$$

$$= \frac{k_1}{r} \qquad = \frac{2}{3}\pi\rho G m r^2$$

$$= \frac{k}{2} r^2$$

$$V'(r) = \frac{k_1}{r} + \frac{k}{2} r^2 + \frac{L^2}{mr^2} \qquad k_1 < 0,\ k < 0$$

$$\frac{dV'}{dr} = \frac{k_1}{r^2} + kr - \frac{L^2}{mr^3}$$

$$= \frac{GMm}{r^2} + \frac{4}{3}\pi\rho G m r - \frac{L^2}{mr_3}$$

For a circular orbit of r_0

$$\frac{L^2}{mr_0^3} = \frac{GMm}{r_0^2} + \frac{4}{3}\pi\rho G m r_0$$

$$L^2 = GMm^2 r_0 + \frac{4}{3}\pi\rho G m^2 r_0^4$$

$$L = \left(GMm^2 r_0 + \frac{4}{3}\pi\rho G m^2 r_0^4\right)^{1/2}$$

$$\dot{\theta} = \frac{L}{mr_0^2}$$

$$\omega_c = \left(\frac{3GMr_0 + 4\pi\rho G r_0^4}{3r_0^4}\right)^{1/2}$$

$$\omega_r^2 = \frac{1}{m}\left(\frac{d^2V'}{dr^2}\right)_{r_0}$$

$$\frac{d^2V'}{dr^2} = \frac{d}{dr}\left(\frac{GMm}{r^2} + \frac{4}{3}\pi\rho G m r - \frac{L^2}{mr_3}\right)$$

$$= -\frac{2GMm}{r^3} + \frac{4}{3}\pi\rho G m + \frac{3L^2}{mr^4}$$

$$\left(\frac{d^2V'}{dr^2}\right)_{r_0} = -\frac{2GMm}{r_0^3} + \frac{4}{3}\pi\rho G m + \frac{3L^2}{mr_0^4}$$

$$\omega_r^2 = \frac{-2GM}{r_0^3} + \frac{4}{3}\pi\rho G + \frac{3}{m^2 r_0^4}\left(GMm^2 r_0 + \frac{4}{3}\pi\rho G m^2 r_0^4\right)$$

$$= \frac{-2GM}{r_0^3} + \frac{4}{3}\pi\rho G + \frac{3GM}{r_0^3} + 4\pi\rho G$$

$$= \frac{GM}{r_0^3} + \frac{16\pi\rho G}{3}$$

$$= \frac{GMr_0 + \frac{16}{3}\pi\rho G r_0^4}{r_0^4}$$

$$\omega_r = \left(\frac{3GMr_0 + 16\pi\rho G r_0^4}{3r_0^4}\right)^{1/2}$$

For F' much less than F, expand ω_c and ω_r

$$\omega_c = \left(\frac{GM}{r_0^3} + \frac{4}{3}\pi\rho G\right)^{1/2} \qquad (a+b)^n = a^{\frac{1}{2}} + \frac{1}{2}a^{-\frac{1}{2}}b + \text{higher orders of } b$$

$$a = \frac{GM}{r_0^3}, \quad b = \frac{4}{3}\pi\rho G$$

$$\omega_c \approx \left(\frac{GM}{r_0^3}\right)^{1/2} + \frac{2}{3}\pi\rho G\left(\frac{r_0^3}{GM}\right)^{1/2}$$

$$\omega_r = \left(\frac{GM}{r_0^3} + \frac{16}{3}\pi\rho G\right)^{1/2}$$

$$a = \frac{GM}{r_0^3}, \quad b = \frac{16}{3}\pi\rho G$$

$$\omega_r \approx \left(\frac{GM}{r_0^3}\right)^{1/2} + \frac{8}{3}\pi\rho G\left(\frac{r_0^3}{GM}\right)^{1/2}$$

$$\omega_p = \omega_r - \omega_c = \left(\frac{8}{3} - \frac{2}{3}\right)\pi\rho\left(\frac{Gr_0^3}{M}\right)^{1/2}$$

$$\boxed{\omega_p = 2\pi\rho\left(\frac{Gr_0^3}{M}\right)^{1/2}}$$

b) Since $\omega_r > \omega_c$, r will reach a maximum before $\theta = 180°$. Therefore, the axis of precession will move in the opposite direction of the orbit.

3-51

a) $F(r) = -\frac{K}{r^2} + \frac{K'}{r^3}$, $K>0$, $K'>0$ or $K'<0$

$V(r) = -\int \frac{K}{r^2}dr - \int \frac{K'}{r^3}dr$, $K<0$

$'V'(r) = \frac{-K}{r} + \frac{K'}{2r^2} + \frac{L^2}{2mr^2} = \frac{-K}{r} + \frac{mK'+L^2}{2mr^2}$

b) $mr^2\dot{\theta} = L$

$\dot{\theta} = \frac{L}{mr^2}$

$r = \frac{1}{u}$ $u = \frac{1}{r}$

$\dot{r} = \frac{-1}{u^2}\frac{du}{dt}$

$= \frac{-1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$

$= \frac{-1}{u^2}\frac{du}{d\theta}\frac{L}{m}u^2$

$\dot{r} = -\frac{L}{m}\frac{du}{d\theta}$

$\ddot{r} = -\frac{L}{m}\frac{d^2u}{d\theta^2}\frac{d\theta}{dt}$

$= -\frac{L}{m}\frac{d^2u}{d\theta^2}\frac{L}{mr^2}$

$= \frac{-L^2}{m^2}u^2\frac{d^2u}{d\theta^2}$

$m\ddot{r} = F(r) + \frac{L^2}{mr^3}$

$\frac{-L^2}{m}u^2\frac{d^2u}{d\theta^2} = F(r) + \frac{L^2}{mr^3}$

$\frac{d^2u}{d\theta^2} = -\frac{m}{L^2u^2}F(\frac{1}{u}) - u$

$F(\frac{1}{u}) = \frac{-K}{(\frac{1}{u})^2} + \frac{K'}{(\frac{1}{u})^3}$

$= Ku^2 + Ku^3$

$\frac{d^2u}{d\theta^2} = -\frac{m}{L^2u^2}\left[Ku^2 + K'u^3\right] - u$

$= \frac{-mK}{L^2} - \frac{mK'u}{L^2} - u$

$\frac{d^2u}{d\theta^2} + u\left(1 + \frac{mK'}{L^2}\right) = -\frac{mK}{L^2}$

$\frac{d^2u}{d\theta^2} + u\left(\frac{L^2+mK'}{L^2}\right) = -\frac{mK}{L^2}$

Homogeneous solution

$\frac{d^2u}{d\theta^2} + u\left(\frac{L^2+mK'}{L^2}\right) = 0$ $u = e^{\lambda\theta}$

$\lambda^2 + \left(\frac{L^2+mK'}{L^2}\right) = 0$

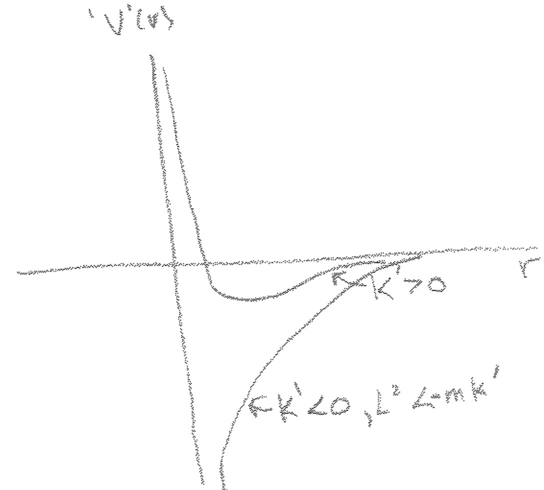
$\lambda^2 = -\left(\frac{L^2+mK'}{L^2}\right)$

$\lambda = \pm i\frac{(L^2+mK')^{1/2}}{L}$

$u_h = A\cos\left(\frac{(L^2+mK')^{1/2}}{L}\theta - \theta_0\right)$

$u_p = -\frac{mK}{L^2}$

$u = \frac{1}{r} = \frac{-mK}{L^2} + A\cos\left(\frac{(L^2+mK')^{1/2}}{L^2}\theta - \theta_0\right)$, $L^2 > -mK'$ for elliptical orbits



$K'<0$: hyperbolic or parabolic orbit

$K'>0$: closed orbit: circular or elliptical about minimum 'V'(r) when $E<0$. unbounded when $E \geq 0$

For $L^2 > -mK'$, $\cos\left(\frac{(L^2+mK')^{1/2}}{L}\theta - \theta_0\right)$ is real

Turning points are

$$\frac{1}{r_1} = \frac{-mK}{L^2} + A \quad , \quad \frac{1}{r_2} = \frac{-mK}{L^2} - A$$

This is equivalent to 3.238 therefore,

$$A^2 = \frac{m^2K^2}{L^4} + \frac{2mE}{L^2} \quad , E < 0$$

$$B = \frac{-mK}{L^2} \quad , K < 0$$

$$\frac{1}{r} = B + A\cos(\alpha\theta - \theta_0) \qquad \alpha = \frac{(L^2+mK')^{1/2}}{L}$$

So, $$r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\alpha\theta} \qquad \epsilon = \frac{A}{|B|} \ , \ a = \left|\frac{B}{A^2-B^2}\right| = \left|\frac{K}{2E}\right|$$

c) $$\omega = \frac{2\pi}{\tau}$$

$$\tau = \frac{2\pi}{\omega}$$

$$\tau_{n.p.} = \frac{2\pi}{\alpha} = 2\pi \quad , \alpha = 1$$

$$\tau = \frac{2\pi}{\alpha}$$

$$\tau_p = \tau - \tau_{n.p.}$$

$$= \frac{2\pi}{\alpha} - 2\pi$$

$$= 2\pi\left(\frac{1}{\alpha} - 1\right)$$

$$\tau_p = 2\pi(1-\alpha)/\alpha \quad , \alpha = \frac{(L^2+mK')^{1/2}}{L}$$

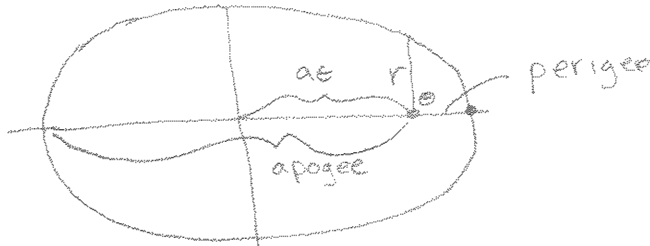
When $\alpha < 1$, $(1-\alpha)/\alpha$ is positive and precession is in direction of $\dot{\theta}$. When $\alpha > 1$, $(1-\alpha)/\alpha$ is negative and precesses in opposite direction.

3-53

d_1 = Perigee = 360 Km + R_e = 360 Km + 6378 Km = 6738 Km

d_2 = Apogee = 2,549 km + R_e = 2549 Km + 6378 Km = 8927 Km

(equatorial)



$2a$ = apogee + perigee

$= d_2 + d_1$

$$a = \frac{d_2 + d_1}{2}$$

$$a\epsilon = a - d_1$$

$$\epsilon = \frac{a - d_1}{a} = 1 - \frac{d_1}{a} = 1 - \frac{d_1}{\frac{d_2 + d_1}{2}} = 1 - \frac{2d_1}{d_2 + d_1}$$

$$= \frac{d_2 + d_1 - 2d_1}{d_1 + d_2}$$

$$= \frac{d_2 - d_1}{d_2 + d_1}$$

At $\theta = 90°$

$$r = \frac{a(1-\epsilon^2)}{1 + \epsilon \cos 90°}$$

$$= \frac{a(1-\epsilon^2)}{1}$$

$$= \frac{d_2 + d_1}{2}\left(1 - \left(\frac{d_2 - d_1}{d_2 + d_1}\right)^2\right)$$

$$= \frac{8927 + 6738}{2}\left(1 - \left(\frac{8927 - 6738}{8927 + 6738}\right)^2\right) \text{Km}$$

$$= \frac{15665}{2}\left(1 - \left(\frac{2189}{15665}\right)^2\right)$$

$$r = 7680 \text{ Km}$$

D = Distance above earth's surface = $r - R_e$

$= 7680 \text{ Km} - 6378 \text{ Km}$

$$\boxed{D = 1302 \text{ Km}}$$

3-55 $$V' = -\frac{\eta m M G R^2}{5r^3}(1-3\cos^2\theta)$$

$$V'(r,\tfrac{\pi}{2}) = -\frac{K_1}{r^3} \quad , \quad K_1 = \frac{\eta m M G R^2}{5}$$

$$V'(r) = \frac{-K}{r} + \frac{L^2}{2mr^2} - \frac{K_1}{r^3} \quad , \quad K = GMm$$

$$0 = \frac{dV'}{dr} = \frac{K}{r^2} - \frac{L^2}{mr^3} + \frac{3K_1}{r^4}$$

$$\frac{L^2}{mr^3} = \frac{K}{r^2} + \frac{3K_1}{r^4}$$

$$L^2 = mKr + \frac{3mK_1}{r}$$

$$L = \left(mKr + \frac{3mK_1}{r}\right)^{1/2}$$

$$\dot{\theta} = \frac{L}{mr^2}$$

$$= \frac{\left(mKr + \frac{3mK_1}{r}\right)^{1/2}}{mr^2}$$

$$\dot{\theta} = \left(\frac{K}{mr^3} + \frac{3K_1}{mr^5}\right)^{1/2}$$

$$\omega_r^2 = \frac{1}{m}\left(\frac{d^2V'}{dr^2}\right)$$

$$= \frac{1}{m}\frac{d}{dr}\left(\frac{K}{r^2} - \frac{L^2}{mr^3} + \frac{3K_1}{r^4}\right)$$

$$= \frac{1}{m}\left(\frac{-2K}{r^3} + \frac{3L^2}{mr^4} - \frac{12K_1}{r^5}\right)$$

$$= \frac{-2K}{mr^3} - \frac{12K_1}{mr^5} + \frac{3}{m^2r^4}\left(mKr + \frac{3mK_1}{r}\right)$$

$$= \frac{-2K}{mr^3} - \frac{12K_1}{mr^5} + \frac{3K}{mr^3} + \frac{9K_1}{mr^5}$$

$$= \frac{K}{mr^3} - \frac{3K_1}{mr^5}$$

$$\omega_r = \left(\frac{K}{mr^3} - \frac{3K_1}{mr^5}\right)^{1/2}$$

$$(x+y)^n = x^n + nx^{n-1}y + \cdots$$

$$\dot{\theta} \doteq \left(\frac{K}{mr^3}\right)^{1/2} + \frac{1}{2}\left(\frac{mr^3}{K}\right)^{1/2}\left(\frac{3K_1}{mr^5}\right) \qquad \omega_r \doteq \left(\frac{K}{mr^3}\right)^{1/2} - \frac{1}{2}\left(\frac{mr^3}{K}\right)^{1/2}\left(\frac{3K_1}{mr^5}\right)$$

$$\omega_p \doteq \dot{\theta} - \omega_r$$

$$\doteq \left(\frac{mr^3}{K}\right)^{1/2}\left(\frac{3K_1}{mr^5}\right)$$

$$\doteq \left(\frac{r^3}{GM}\right)^{1/2}\left(\frac{\frac{3}{5}\eta GMR^2}{r^5}\right)$$

$$\doteq \frac{3}{5}\eta\frac{R^2}{r^2}\left(\frac{GMr^3}{r^6}\right)^{1/2}$$

$$\boxed{\omega_p \doteq \frac{3}{5}\eta\left(\frac{R}{r}\right)^2\left(\frac{GM}{r^3}\right)^{1/2}}$$

3-57

$$V_{rel} = V(r) - \frac{[E - V(r)]^2}{2mc^2}$$

$$V(r) = \frac{K}{r}$$

$$E = \frac{1}{2} m\dot{r}^2 + \frac{L^2}{2mr^2} + V(r)$$

$$V'_{rel}(r) = V_{rel}(r) + \frac{L^2}{2mr^2}$$

$$V'_{rel} = V - \frac{[E-V]^2}{2mc^2} + \frac{L^2}{2mr^2}$$

$$\frac{dV'_{rel}}{dr} = \frac{dV}{dr} - \frac{[E-V]}{mc^2}\left(\frac{dE}{dr} - \frac{dV}{dr}\right) - \frac{L^2}{mr^3}$$

$$= \frac{dV}{dr} - \frac{[E-V]}{mc^2}\left(-\frac{dV}{dr}\right) - \frac{L^2}{mr^3} \quad , E = \text{constant}$$

$$= \frac{dV}{dr}\left(1 + \frac{[E-V]}{mc^2}\right) - \frac{L^2}{mr^3}$$

$$= \frac{-K}{r^2}\left(1 + \frac{L^2}{2m^2r^2c^2}\right) - \frac{L^2}{mr^3} \quad (\dot{r} = 0)$$

$$= \frac{-K}{r^2} - \left(\frac{K}{2m^2r^4c^2} + \frac{1}{mr^3}\right)L^2$$

$$L^2\left(\frac{K}{2m^2r^4c^2} + \frac{1}{mr^3}\right) = \frac{-K}{r^2}$$

$$\frac{L^2}{mr^3}\left(\frac{K}{2mrc^2} + 1\right) = \frac{-K}{r^2}$$

$$L^2 = Kmr\left(1 - \frac{K}{2mrc^2}\right)^{-1} \quad K < 0 \text{ (attractive force)}$$

$$L = (Kmr)^{1/2}\left(1 - \frac{K}{2mrc^2}\right)^{-1/2}$$

$$\dot{\theta} = \frac{L}{mr^2}$$

$$= \left(\frac{K}{mr^3}\right)^{1/2}\left(1 - \frac{K}{2mrc^2}\right)^{-1/2} \quad (1-x)^{-1/2} \doteq 1 + \frac{1}{2}x$$

$$\dot{\theta} \doteq \left(\frac{K}{mr^3}\right)^{1/2}\left(1 + \frac{1}{2}\frac{K}{2mrc^2}\right)$$

$$\doteq \left(\frac{K}{mr^3}\right)^{1/2} + \left(\frac{K}{mr^3}\right)^{1/2}\left(\frac{K}{4mrc^2}\right)$$

$$\dot{\theta} \doteq \left(\frac{K}{mr^3}\right)^{1/2} + \frac{K^{3/2}}{4m^{3/2}r^{5/2}c^2}$$

$$\frac{d^2V_{rel}}{dr^2} = \frac{d}{dr}\left(\frac{K}{r^2} + \frac{KL^2}{2m^2r^4c^2} - \frac{L^2}{mr^3}\right), \quad K<0$$

$$= \frac{-2K}{r^3} - \frac{2KL^2}{m^2r^5c^2} + \frac{3L^2}{mr^4}$$

$$\omega_r^2 = \frac{-2K}{mr^3} - \frac{2KL^2}{m^3r^5c^2} + \frac{3L^2}{m^2r^4}$$

$$= \frac{-2K}{mr^3} - \frac{2K}{m^3r^5c^2}\left(Kmr\left(1-\frac{K}{2mrc^2}\right)^{-1}\right) + \frac{3}{m^2r^4}\left(Kmr\left(1-\frac{K}{2mrc^2}\right)^{-1}\right)$$

$$= \frac{-2K}{mr^3} - \frac{2K^2}{m^2r^4c^2}\left(1-\frac{K}{2mrc^2}\right)^{-1} + \frac{3K}{mr^3}\left(1-\frac{K}{2mrc^2}\right)^{-1}$$

$$= \frac{-2K}{mr^3} + \left[\frac{3K}{mr^3} - \frac{2K^2}{m^2r^4c^2}\right]\left(1-\frac{K}{2mrc^2}\right)^{-1}$$

$$= \frac{\frac{-2K}{mr^3}\left(1-\frac{K}{2mrc^2}\right) + \frac{3K}{mr^3} - \frac{2K^2}{m^2r^4c^2}}{1-\frac{K}{2mrc^2}}$$

$$= \frac{\frac{K}{mr^3} + \frac{2K^2}{2m^2r^4c^2} - \frac{2K^2}{m^2r^4c^2}}{1-\frac{K}{2mrc^2}}$$

$$= \left(\frac{K}{mr^3} - \frac{K^2}{m^2r^4c^2}\right)\left(1-\frac{K}{2mrc^2}\right)^{-1}$$

$$\omega_r = \left(\frac{K}{mr^3} - \frac{K^2}{m^2r^4c^2}\right)^{1/2}\left(1-\frac{K}{2mrc^2}\right)^{-1/2}$$

$$= \left(\frac{K}{mr^3}\left(1-\frac{K}{mrc^2}\right)\right)^{1/2}\left(1-\frac{K}{2mrc^2}\right)^{-1/2}$$

$$= \left(\frac{K}{mr^3}\right)^{1/2}\left(1-\frac{K}{mrc^2}\right)^{1/2}\left(1-\frac{K}{2mrc^2}\right)^{-1/2} \qquad (1-x)^{-1/2} \doteq 1+\tfrac{1}{2}x$$

$$\doteq \left(\frac{K}{mr^3}\right)^{1/2}\left(1-\frac{K}{2mrc^2}\right)\left(1+\frac{K}{4mrc^2}\right) \qquad (1-x)^{1/2} \doteq 1-\tfrac{1}{2}x$$

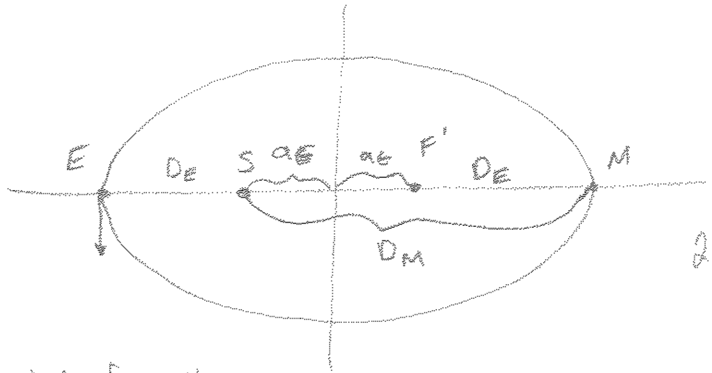
$$\doteq \left(\frac{K}{mr^3}\right)^{1/2}\left(1-\frac{K}{4mrc^2} - \frac{K^2}{8(mrc^2)^2}\right)$$

$$\doteq \left(\frac{K}{mr^3}\right)^{1/2}\left(1-\frac{K}{4mrc^2}\right)$$

$$\omega_r \doteq \left(\frac{K}{mr^3}\right)^{1/2} - \frac{K^{3/2}}{4m^{3/2}r^{5/2}c^2}$$

$$\boxed{\omega_p = \dot{\theta} - \omega_r \doteq \frac{1}{2}K^{3/2}m^{-3/2}r^{-5/2}c^{-2}}$$

3-59



$$2a = D_m + D_E$$

$$a = \frac{D_m + D_E}{2}$$

$$2a\epsilon = D_m - D_E$$

$$a\epsilon = \frac{D_m - D_E}{2}$$

$$\epsilon = \frac{D_m - D_E}{2a}$$

$$\epsilon = \frac{D_m - D_E}{D_m + D_E}$$

$$b = a(1-\epsilon^2)^{1/2}$$

$$= \frac{D_m + D_E}{2}\left(1 - \left(\frac{D_m - D_E}{D_m + D_E}\right)^2\right)^{1/2}$$

$$= \frac{D_m + D_E}{2}\left(\frac{(D_m + D_E)^2 - (D_m - D_E)^2}{(D_m + D_E)^2}\right)^{1/2}$$

$$b = \frac{D_m + D_E}{2}\left(\frac{D_m^2 + 2D_mD_E + D_E^2 - (D_m^2 - 2D_mD_E + D_E^2)}{(D_m + D_E)^2}\right)^{1/2}$$

$$= \frac{D_m + D_E}{2}\left(\frac{4D_mD_E}{(D_m + D_E)^2}\right)^{1/2}$$

$$b = \sqrt{D_mD_E}$$

For circular orbit of earth,

$$\omega_E = \frac{2\pi}{1\,\text{year}}$$

$$= 2.00 \times 10^{-7}\,s^{-1}$$

$$L_s = m(\omega_E + \omega_s)D_E^2$$

$$\tau_s = \frac{2m}{L_s}\pi ab$$

$$= \frac{2m}{m(\omega_E + \omega_s)D_E^2}\pi ab$$

$$= \frac{2\pi}{(\omega_E + \omega_s)D_E^2}\left(\frac{D_m + D_E}{2}\right)(D_mD_E)^{1/2}$$

$$= \frac{\pi(D_m + D_E)}{(\omega_E + \omega_s)D_E^2}(D_mD_E)^{1/2}$$

$$\tau_s^2 = \frac{4\pi^2}{MG}a^3 \quad (\text{Kepler's Law})$$

$$\tau_s = \frac{2\pi}{(MG)^{1/2}}a^{3/2} = 1.37\,\text{years}$$

$$\frac{2\pi}{(MG)^{1/2}}\left(\frac{D_m + D_E}{2}\right)^{3/2} = \frac{\pi(D_m + D_E)}{(\omega_E + \omega_s)D_E^2}(D_mD_E)^{1/2}$$

$$\omega_E + \omega_s = 2.188 \times 10^{-7}/s$$

$$\omega_s = .188 \times 10^{-7}/s$$

$$V_s = \omega_s D_E$$

$$\boxed{V_s = 2.8\,\text{Km/s}}$$

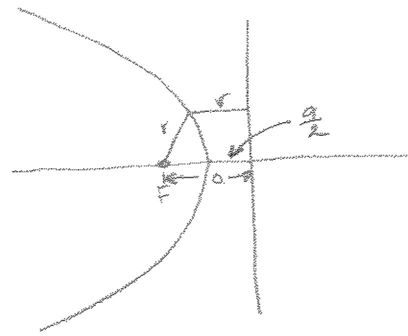
$$G = 6.67 \times 10^{-11}\,m^3\,Kg^{-1}\,s^{-2}$$

$$M = 1.99 \times 10^{30}\,Kg$$

$$D_E = 1.49 \times 10^{11}\,m$$

$$D_M = 2.2 \times 10^{11}\,m$$

To escape from sun's gravitational pull, orbit must be a parabola.



$$\frac{1}{r} = B + A\cos\theta$$

$B = A$ for parabola

$$B = \frac{1}{a}, \quad A = \frac{1}{a}$$

For inverse square law force

$$B = \frac{-mK}{L^2} \quad (3.259)$$

$$\frac{1}{a} = \frac{-mK}{L^2}$$

$$= \frac{-mK}{(mD_E^2\omega)^2} \qquad K = -GMm$$

$$= \frac{GMm^2}{m^2\omega^2 D_E^4}$$

$$\omega^2 = \frac{GMa}{D_E^4}$$

$$\omega = \frac{(GMa)^{1/2}}{D_E^2}$$

$\frac{a}{2} = D_E \rightarrow a = 2D_E$ for parabola

$G = 6.67 \times 10^{-11}\, m^3 Kg^{-1} s^{-2}$

$M = 1.99 \times 10^{30}\, Kg$

$D_E = 1.49 \times 10^{11}\, m$

$\omega_e = 2.00 \times 10^{-7}\, s^{-1}$

$$\omega = 2.833 \times 10^{-7}\, s^{-1}$$

$$\omega_e + \omega_s = 2.833 \times 10^{-7}\, s^{-1}$$

$$\omega_s = 0.833 \times 10^{-7}\, s^{-1}$$

$$V_{esc} = \omega_s D_E$$

$$\boxed{V_{esc} = 12.3\, Km/s}$$

5-61 The probe velocity v_1 is relative to Planet 1 velocity.

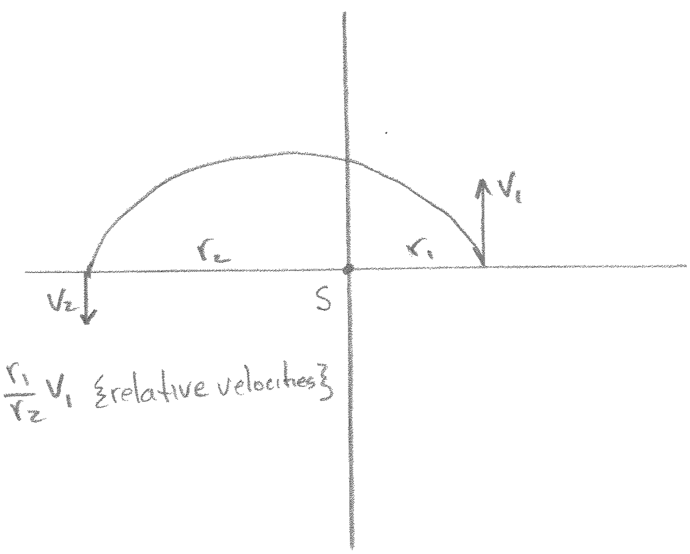
The relative velocity at a distance r_2 is found from the relation

$$L_1 = m r_2 v_{\perp 2} \quad \text{(conservation of angular momentum)}$$

$$m r_1 v_1 = m r_2 v_{\perp 2}$$

$$v_{\perp 2} = \frac{r_1}{r_2} v_1$$

where $v_{\perp 2}$ is the component of v_2 perpendicular to the radius vector $\vec{r}$.
It's evident v_2 should be a minimum at r_2. This occurs when $v_2 = v_{\perp 2}$ (no radial component). This only occurs for an ellipse with an aphelion at r_2. {Assume no gravitational effects other than Sun-probe.}



$$v_2 = \frac{r_1}{r_2} v_1 \quad \text{\{relative velocities\}}$$

Therefore, the semimajor axis is

$$a = \frac{r_1 + r_2}{2}$$

Period of Planet 1

$$F_{cent} = F_g$$

$$\frac{m r_1^2 \omega_1^2}{r_1} = \frac{G m M}{r_1^2}$$

$$\omega_1^2 = \frac{GM}{r_1^3}$$

$$\omega_1 = \sqrt{\frac{GM}{r_1^3}}$$

$$T_1 = \frac{2\pi}{\omega_1}$$

$$T_1 = \frac{2\pi r_1^{3/2}}{\sqrt{GM}} \longrightarrow T_2 = \frac{2\pi r_2^{3/2}}{\sqrt{GM}}$$

Planet velocities:

$$v_{p_1} = \frac{2\pi r_1}{T_1} \qquad\qquad v_{p_2} = \frac{2\pi r_2}{T_2}$$

$$= \frac{2\pi r_1}{\frac{2\pi r_1^{3/2}}{\sqrt{GM}}} \qquad\qquad = \sqrt{GM}\, r_2^{-1/2}$$

$$= v_{p_1} r_1^{1/2} r_2^{-1/2} \qquad \left\{ \sqrt{GM} = v_{p_1} r_1^{1/2} \right\}$$

$$= \sqrt{GM}\, r_1^{-1/2} \qquad\qquad v_{p_2} = v_{p_1} \left(\frac{r_1}{r_2}\right)^{1/2}$$

$$v_{p_1} = \sqrt{\frac{GM}{r_1}}$$

Now,

$$a = \left|\frac{K}{2E}\right|$$

$$a = \frac{r_1 + r_2}{2}$$

$$E = \frac{K}{r_1} + \frac{1}{2} m v^2 \ , \ v = v_{p_1} + v_1$$

$$2E = \frac{K}{a}$$

$$\frac{2K}{r_1} + mv^2 = \frac{K}{a}$$

$$mv^2 = \frac{K}{a} - \frac{2K}{r_1}$$

$$= \frac{K r_1 - 2Ka}{r_1 a}$$

$$= \frac{K}{r_1} \frac{(r_1 - 2a)}{a}$$

$$v^2 = \frac{K}{m r_1} \left(\frac{r_1 - 2a}{a}\right)$$

$$= \frac{-GmM}{r_1} \left(\frac{r_1 - 2a}{a}\right)$$

$$= \frac{GM}{r_1}\left(\frac{2a-r_1}{a}\right)$$

$$= \frac{GM}{r_1}\left(\frac{\frac{2(r_2+r_1)}{2} - r}{\frac{r_1+r_2}{2}}\right)$$

$$= V_{P_1}^2\left(\frac{r_2+r_1-r_1}{\frac{r_1+r_2}{2}}\right)$$

$$= V_{P_1}^2\left(\frac{2r_2}{r_1+r_2}\right)$$

$$V = V_{P_1}\left(\frac{2r_2}{r_1+r_2}\right)^{1/2}$$

$$V_1 + V_{P_1} = V_{P_1}\left(\frac{2r_2}{r_1+r_2}\right)^{1/2}$$

$$V_1 = V_{P_1}\left[\left(\frac{2r_2}{r_1+r_2}\right)^{1/2} - 1\right]$$

$$\boxed{V_1 = \frac{2\pi r_1}{T_1}\left[\left(\frac{2r_2}{r_1+r_2}\right)^{1/2} - 1\right]}$$

$$E = \frac{K}{r_2} + \frac{1}{2}mV^2, \quad V = V_{P_2} + V_2$$

$$V^2 = \frac{GM}{r_2}\left(\frac{2a-r_2}{a}\right) \quad \{\text{using method for } V_1\}$$

$$= \frac{GM}{r_2}\left(\frac{r_2+r_1-r_2}{\frac{r_1+r_2}{2}}\right)$$

$$= V_{P_2}^2 \frac{2r_1}{r_1+r_2}$$

$$V = V_{P_2}\left(\frac{2r_1}{r_1+r_2}\right)^{1/2}$$

$$V_2 + V_{P_2} = V_{P_2}\left(\frac{2r_1}{r_1+r_2}\right)^{1/2}$$

$$V_2 = V_{P_2}\left[\left(\frac{2r_1}{r_1+r_2}\right)^{1/2} - 1\right]$$

$$\boxed{V_2 = \frac{2\pi r_1}{T_1}\left(\frac{r_1}{r_2}\right)^{1/2}\left[\left(\frac{2r_1}{r_1+r_2}\right)^{1/2} - 1\right]}$$

Sun-Earth distance = 9.3×10^7 mi.

Sun-Venus distance = 6.7×10^7 mi.

Sun-Mars distance = 1.4×10^8 mi

Earth period = 8760 hrs.

Earth-Venus trip:

$$V_1 = \frac{2\pi r_1}{T_1}\left[\left(\frac{2r_2}{r_1+r_2}\right)^{1/2} - 1\right]$$

$$= \frac{2\pi(9.3\times10^7)}{8760}\left[\left(\frac{2(6.7\times10^7)}{6.7\times10^7+9.3\times10^7}\right)^{1/2} - 1\right]$$

$$\boxed{V_1 = -7.5\times10^3\ \frac{mi}{hr}}$$

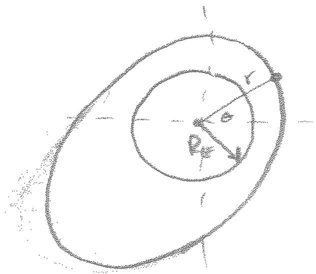
Earth-Mars trip:

$$V_1 = \frac{2\pi r_1}{T_1}\left[\left(\frac{2r_2}{r_1+r_2}\right)^{1/2} - 1\right]$$

$$= \frac{2\pi(9.3\times10^7)}{8760}\left[\left(\frac{2(1.4\times10^8)}{1.4\times10^8+9.3\times10^7}\right)^{1/2} - 1\right]$$

$$\boxed{V_1 = 6.4\times10^3\ \frac{mi}{hr}}$$

3-63



$$\dot{\theta} = \frac{L}{mr^2}$$

i) The perigee has r as a minimum. So, the perigee as a function of θ is located where $\dot{\theta}$ = maximum.

ii) Kepler's Law states

$$\tau^2 = \frac{4\pi^2}{MG} a^3$$

where $M = M_e$. M_eG is to be expressed in terms of g and R_e.

$$F = \frac{GmM}{r^2}$$

$$mg = \frac{GmM}{R_e^2}$$

$$g = \frac{GM}{R_e^2}$$

$$GM = R_e^2 g$$

So,

$$\tau^2 = \frac{4\pi^2}{gR_e^2} a^3$$

$$a^3 = \frac{\tau^2 g R_e^2}{4\pi^2}$$

$$\boxed{a = \left(\frac{\tau^2 g R_e^2}{4\pi^2}\right)^{1/3}}$$

iii) The eccentricity is related to r by

$$r_{1,2} = a(1 \pm \epsilon)$$

$$r_1 = r_{max}, \quad r_2 = r_{min}$$

$$r_{max} = a(1+\epsilon), \quad r_{min} = a(1-\epsilon)$$

$$\frac{r_{max}}{r_{min}} = \frac{1+\epsilon}{1-\epsilon}$$

Let $\alpha = \frac{r_{max}}{r_{min}}$

$$\alpha = \frac{1+\epsilon}{1-\epsilon}$$

$$\alpha - \alpha\epsilon = 1 + \epsilon$$

$$\alpha - 1 = \epsilon + \alpha\epsilon$$

$$\alpha - 1 = \epsilon(1+\alpha)$$

$$\frac{\alpha-1}{\alpha+1} = \epsilon$$

$$L = mr^2\dot{\theta}$$

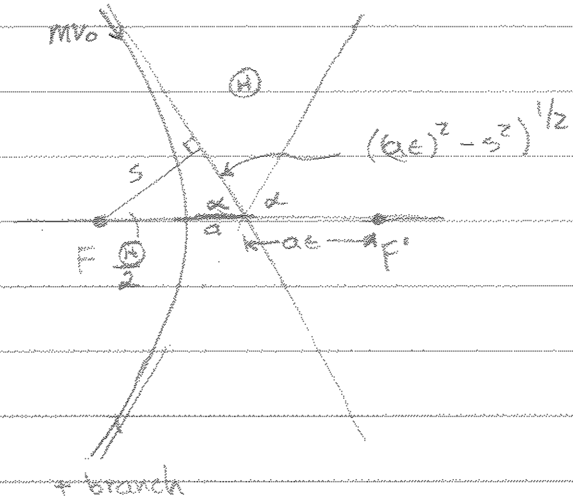
$$r_{max}^2 \dot{\theta}_{min} = r_{min}^2 \dot{\theta}_{max}$$

$$\frac{r_{max}^2}{r_{min}^2} = \frac{\dot{\theta}_{max}}{\dot{\theta}_{min}}$$

$$\frac{r_{max}}{r_{min}} = \left(\frac{\dot{\theta}_{max}}{\dot{\theta}_{min}}\right)^{1/2}$$

$$\boxed{\epsilon = \frac{\alpha-1}{\alpha+1}, \quad \alpha = \left(\frac{\dot{\theta}_{max}}{\dot{\theta}_{min}}\right)^{1/2}}$$

3-65



$$r = \frac{a(\epsilon^2 - 1)}{1 + \epsilon\cos\theta}, \quad \cos\alpha = -\frac{1}{\epsilon}$$

$$r_1 = a(\epsilon - 1)$$

$$r_1 = \frac{K}{2E} + \left(\left(\frac{K}{2E}\right)^2 + \frac{L^2}{2mE}\right)^{1/2}, \quad E > 0,\ K < 0$$

$$= -a + \left(a^2 + \frac{m^2 v_0^2 s^2}{m^2 v_0^2}\right)^{1/2} \qquad E = \frac{1}{2} m v_0^2$$

$$= -a + (a^2 + s^2)^{1/2} \qquad L = m v_0 s$$

$$a(\epsilon - 1) = -a + (a^2 + s^2)^{1/2}$$

$$a + a(\epsilon - 1) = (a^2 + s^2)^{1/2}$$

$$a(1 + \epsilon - 1) = (a^2 + s^2)^{1/2}$$

$$a\epsilon = (a^2 + s^2)^{1/2}$$

$$\epsilon = \left(1 + \frac{s^2}{a^2}\right)^{1/2}$$

$$\tan\frac{\Theta}{2} = \cot\alpha = \frac{((a\epsilon)^2 - s^2)^{1/2}}{s} = \frac{a}{s} = \left|\frac{K}{2E}\right|\frac{1}{s} = \frac{|K|}{m v_0^2 s}$$

$$K = -q_1 q_2$$

$$|K| = |-q_1 q_2| = q_1 q_2$$

$$\tan\frac{\Theta}{2} = \frac{q_1 q_2}{m v_0^2 s}, \quad s = \frac{q_1 q_2}{m v_0^2}\cot\frac{\Theta}{2}$$

$$\frac{1}{2}\sec^2\frac{\Theta}{2}\, d\Theta = \frac{-q_1 q_2}{m v_0^2 s^2}\, ds$$

$$d\sigma = 2\pi s\, ds$$

$$= 2\pi\left[\frac{q_1 q_2}{m v_0^2}\cot\frac{\Theta}{2}\right]\left[\frac{1}{2}\sec^2\frac{\Theta}{2}\,\frac{m v_0^2}{q_1 q_2}\left(\frac{q_1 q_2}{m v_0^2}\cot\frac{\Theta}{2}\right)^2\right]$$

$$= \pi\left[\cot\frac{\Theta}{2}\sec^2\frac{\Theta}{2}\left(\frac{q_1 q_2}{m v_0^2}\right)\cot^2\frac{\Theta}{2}\right]$$

$$= \pi\left[\frac{\cos^3\Theta/2}{\sin^3\Theta/2}\,\frac{1}{\cos^2\Theta/2}\right]\left[\left(\frac{q_1 q_2}{m v_0^2}\right)^2\right]$$

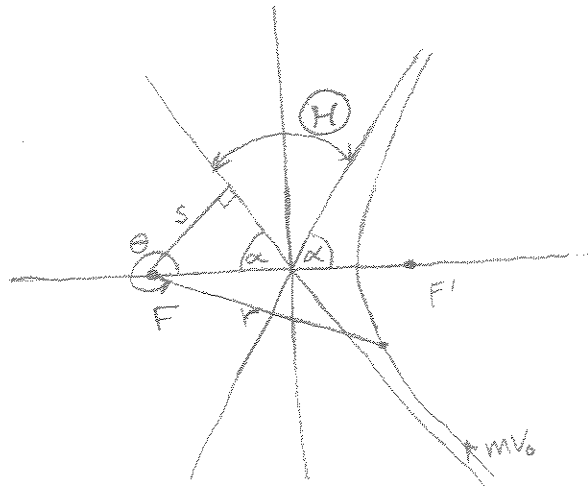
$$= \pi\left(\frac{q_1 q_2}{m v_0^2}\right)^2\frac{\cos\Theta/2}{\sin^3\Theta/2}\cdot\frac{\sin\Theta/2}{\sin\Theta/2}$$

$$= \pi\left(\frac{q_1 q_2}{m v_0^2}\right)^2\frac{\frac{1}{2}\sin\Theta}{\sin^4\Theta/2}$$

$$d\sigma = \left(\frac{q_1 q_2}{2 m v_0^2}\right)^2\frac{2\pi\sin\Theta}{\sin^4\Theta/2}\,,\quad K<0$$

$$\boxed{d\sigma_{K<0} = d\sigma_{K>0}}$$

3-67



$$L = mv_0 s \qquad E = \tfrac{1}{2}mv_0^2$$

$$'V'(r) = V(r) + \frac{L^2}{2mr^2} = E$$

$$= V(r) + \frac{m^2 v_0^2 s^2}{2mr^2}, \quad r = \frac{1}{u}$$

$$'V'\left(\tfrac{1}{u}\right) = V\left(\tfrac{1}{u}\right) + \frac{m^2 v_0^2 s^2}{2m\left(\frac{1}{u}\right)^2}, \quad u = \frac{1}{r}$$

$$'V'\left(\tfrac{1}{u}\right) = V\left(\tfrac{1}{u}\right) + \tfrac{1}{2}mv_0^2 s^2 u^2$$

$$Ⓗ = \pi - 2\alpha$$

$$\dot r = \sqrt{\frac{2}{m}}\left(E - 'V'(r)\right)^{1/2}$$

$$= \sqrt{\frac{2}{m}}\left(\tfrac{1}{2}mv_0^2 - \tfrac{1}{2}mv_0^2 s^2 u^2 - V\left(\tfrac{1}{u}\right)\right)^{1/2}$$

$$= v_0\left(1 - s^2u^2 - \frac{V\left(\frac{1}{u}\right)}{\frac{1}{2}mv_0^2}\right)^{1/2}$$

$$\frac{1}{\dot r} = \frac{1}{v_0}\left(1 - s^2u^2 - \frac{V\left(\frac{1}{u}\right)}{\frac{1}{2}mv_0^2}\right)^{-1/2}$$

$$-\frac{d\theta}{du}\frac{1}{v_0 s} = \frac{1}{v_0}\left(1 - s^2u^2 - \frac{V\left(\frac{1}{u}\right)}{\frac{1}{2}mv_0^2}\right)^{-1/2}$$

$$-d\theta = s\left(1 - s^2u^2 - \frac{V(u^{-1})}{\frac{1}{2}mv_0^2}\right)^{-1/2}du$$

$$\theta = s\int_0^{u_0}\left(1 - s^2u^2 - \frac{V(u^{-1})}{\frac{1}{2}mv_0^2}\right)^{-1/2}du$$

$$r = \frac{1}{u}$$

$$\dot r = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt}$$

$$= -\frac{L}{m}\frac{du}{d\theta}$$

$$= -\frac{mv_0 s}{m}\frac{du}{d\theta}$$

$$= -v_0 s\frac{du}{d\theta}$$

$$\frac{1}{\dot r} = -\frac{d\theta}{du}\frac{1}{v_0 s}$$

But $\theta = 2\alpha$ when r begins at ∞ and ends at r_0 to establish 2α.

$$\boxed{Ⓗ = \left|\pi - 2s\int_0^{u_0}\left(1 - s^2u^2 - V(u^{-1})/\tfrac{1}{2}mv_0^2\right)^{-1/2}du\right| \quad \{Ⓗ \text{ must be positive}\}}$$

u_0 = first positive root of denominator of integrand.

3-69

$$F = \frac{K}{r^2} \quad , K > 0$$

$$V = -\int_{\infty}^{r} \frac{K}{r^2} dr$$

$$= \frac{K}{r}$$

$$\Theta = \left| \pi - 2s \int_0^{u_0} \left[1 - s^2u^2 - V(u^{-1}) / \tfrac{1}{2} m v_0^2 \right]^{-\frac{1}{2}} du \right|$$

$$= \left| \pi - 2s \int_0^{u_0} \left[1 - s^2u^2 - Ku / \tfrac{1}{2} m v_0^2 \right]^{-\frac{1}{2}} du \right|$$

Completing the square:

$$-s^2u^2 - Ku/\tfrac{1}{2}mv_0^2 + 1 = -s^2\left(u^2 + \frac{2Ku}{mv_0^2 s^2}\right) + 1$$

$$= -s^2\left(u^2 + \frac{2Ku}{mv_0^2s^2} + \left(\frac{K}{mv_0^2s^2}\right)^2\right) + 1 + \left(\frac{Ks}{mv_0^2s^2}\right)^2$$

$$= -s^2\left(u + \frac{K}{mv_0^2s^2}\right)^2 + \frac{(mv_0^2s^2)^2 + (Ks)^2}{(mv_0^2s^2)^2}$$

$$= -s^2(x)^2 + B$$

$$= B - s^2x^2$$

$$-s^2\left(u_0 + \frac{K}{mv_0^2s^2}\right)^2 + B = 0$$

$$u_0 + \frac{K}{mv_0^2s^2} = \pm\left(\frac{B}{s^2}\right)^{1/2}$$

$$u_0 = \frac{-K}{mv_0^2s^2} + \frac{B^{1/2}}{s}$$

$$\Theta = \left| \pi - 2s \int_{x_1}^{x_2} \left[B - s^2x^2 \right]^{-\frac{1}{2}} dx \right| \quad \left\{ x = u + \frac{K}{mv_0^2s^2} \right\}$$

let $B\sin^2\theta = s^2x^2$

$$= \left| \pi - 2s\frac{B^{1/2}}{s} \int_{\theta_1}^{\theta_2} \frac{\cos\theta d\theta}{\left[B - B\sin^2\theta\right]^{1/2}} \right| \qquad sx = B^{1/2}\sin\theta$$

$$x = \frac{B^{1/2}}{s}\sin\theta$$

$$= \left| \pi - 2\int_{\theta_1}^{\theta_2} d\theta \right| \qquad dx = \frac{B^{1/2}}{s}\cos\theta d\theta$$

$$= \left| \pi - 2\theta \Big|_{\theta_1}^{\theta_2} \right|$$

$$= \left| \pi - 2\sin^{-1}\left(x\frac{s}{B^{1/2}}\right) \Big|_{x_1}^{x_2} \right|$$

$$= \left| \pi - 2\sin^{-1}\left(\left[u + \frac{K}{mv_0^2 s^2}\right]\frac{s}{B^{1/2}}\right)\Big|_0^{u_0}\right|$$

$$= \left|\pi - 2\left[\sin^{-1}\left(\frac{B^{1/2}}{s}\cdot\frac{s}{B^{1/2}}\right) - \sin^{-1}\left(\frac{Ks}{mv_0^2 s^2 B^{1/2}}\right)\right]\right|$$

$$= \left|\pi - 2\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{Ks}{mv_0^2 s^2 B^{1/2}}\right)\right)\right|$$

$$= 2\sin^{-1}\left(\frac{Ks}{mv_0^2 s^2 B^{1/2}}\right)$$

$$\sin\frac{\Theta}{2} = \frac{Ks}{mv_0^2 s^2 B^{1/2}} \qquad B = \frac{(mv_0^2 s^2)^2 + (Ks)^2}{(mv_0^2 s^2)^2}$$

$$= \frac{Ks\, mv_0^2 s^2}{mv_0^2 s^2\left((mv_0^2 s^2)^2 + (Ks)^2\right)^{1/2}}$$

$$= \frac{Ks}{\left[(mv_0^2 s^2)^2 + (Ks)^2\right]^{1/2}}$$

$$= \frac{K}{\left[(mv_0^2 s)^2 + K\right]^{1/2}}$$

$$\sin^2\frac{\Theta}{2} = \frac{K^2}{(mv_0^2 s)^2 + K}$$

$$\frac{1}{\sin^2\frac{\Theta}{2}} = \frac{(mv_0^2 s)^2 + K}{K^2}$$

$$\frac{K^2}{\sin^2\frac{\Theta}{2}} - K = (mv_0^2 s)^2$$

$$\frac{\left(K^2\csc^2\Theta/2 - K\right)^{1/2}}{mv_0^2} = s$$

$$ds = \frac{1}{2mv_0^2}\left(K^2\csc^2\Theta/2 - K\right)^{-1/2} d\left(K^2\csc^2\Theta/2\right) \qquad d\left(\sin\frac{\Theta}{2}\right)^{-2} = -2\left(\sin\frac{\Theta}{2}\right)^{-3}\cos\frac{\Theta}{2}\,\frac{d\Theta}{2}$$

$$= -\frac{\cos\left(\frac{\Theta}{2}\right)d\Theta}{\sin^3\frac{\Theta}{2}}$$

$$ds = \frac{1}{2mv_0^2}\left(K^2\csc^2\frac{\Theta}{2} - K\right)^{-1/2}\left(\frac{-\cos\left(\frac{\Theta}{2}\right)}{\sin^3\frac{\Theta}{2}}\right)d\Theta$$

$$= \frac{-K^2\cos\Theta/2}{2mv_0^2\left(K^2\csc^2\frac{\Theta}{2} - K\right)^{1/2}\sin^3\frac{\Theta}{2}}\,d\Theta$$

$$d\sigma = 2\pi s\,ds$$

$$= 2\pi\left[\frac{\left(K^2\csc^2\Theta - K\right)^{1/2}}{mv_0^2}\right]\left[\frac{K^2\cos\Theta/2}{2mv_0^2\left(K^2\csc^2\frac{\Theta}{2} - K\right)^{1/2}\sin^3\frac{\Theta}{2}}\right]d\Theta$$

$$= 2\pi\,\frac{K^2\cos\Theta/2}{2(mv_0^2)^2\sin^3\frac{\Theta}{2}}\,d\Theta$$

$$= \frac{2\pi K^2\cos\Theta/2\,\sin\Theta/2}{(mv_0^2)^2\sin^3\frac{\Theta}{2}\,\sin\frac{\Theta}{2}}\,d\Theta$$

$$= \frac{2\pi K^2}{2(mv_0^2)^2}\,\frac{\frac{1}{2}\sin\Theta}{\sin^4\frac{\Theta}{2}}\,d\Theta$$

$$\boxed{d\sigma = \left(\frac{K}{2mv_0^2}\right)^2\frac{2\pi\sin\Theta}{\sin^4\frac{\Theta}{2}}\,d\Theta}$$

3-71

$$F(r) = \frac{K}{r^3}, \quad K > 0$$

$$\frac{d^2u}{d\theta^2} = -u - \frac{m}{L^2u^2}F\left(\frac{1}{u}\right)$$

$$= -u - \frac{m}{L^2u^2}Ku^3$$

$$= -u - \frac{mK}{L^2}u$$

$$\frac{d^2u}{d\theta^2} + u\left(1 + \frac{mK}{L^2}\right) = 0 \quad , \quad u = e^{\lambda\theta}$$

$$\lambda^2 + 1 + \frac{mK}{L^2} = 0$$

$$\lambda^2 = -\left(1 + \frac{mK}{L^2}\right)$$

$$\lambda = \pm i\left(1 + \frac{mK}{L^2}\right)^{1/2}$$

$$u = A\cos\left[\left(1 + \frac{mK}{L^2}\right)^{1/2}(\theta - \theta_0)\right]$$

$$V'(r) = \frac{K}{2r^2} + \frac{L^2}{2mr^2} = E$$

$$\frac{1}{r^2}\left(\frac{L^2 + Km}{2m}\right) = E$$

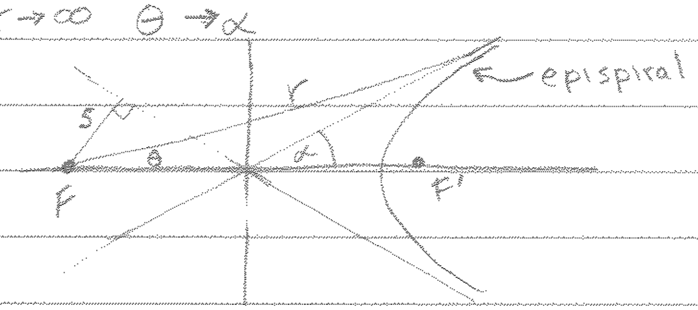
$$\frac{1}{r^2} = \frac{2mE}{L^2 + Km} \quad \{\text{turning point}\}$$

$$\boxed{\frac{1}{r} = \left(\frac{2mE}{L^2 + Km}\right)^{1/2}\cos\left[\left(1 + \frac{mK}{L^2}\right)^{1/2}(\theta - \theta_0)\right]}$$

Since $\left(1 + \frac{mK}{L^2}\right)^{1/2} > 1$, $\theta - \theta_0$ range must be small so that $\frac{1}{r} > 0$ In particular,

$$0 \le \left(1 + \frac{mK}{L^2}\right)^{1/2}(\theta - \theta_0) \le \frac{\pi}{2} \text{ and } \frac{3\pi}{2} \le \left(1 + \frac{mK}{L^2}\right)^{1/2}(\theta - \theta_0) \le 2\pi$$

As $r \to \infty$ $\theta \to \alpha$



When $r \to \infty$, $\left(1+\frac{mk}{L^2}\right)^{1/2}(\theta) \to \frac{\pi}{2}$ $\{\theta_0 = 0\}$

$$\theta = \frac{\pi}{2}\left(1+\frac{mk}{L^2}\right)^{-1/2}$$

But $\theta \to \alpha$ as $r \to \infty$

$$\alpha = \frac{\pi}{2}\left(1+\frac{mk}{L^2}\right)^{-1/2}$$

$$\Theta = \pi - 2\alpha$$

$$= \pi - \pi\left(1+\frac{mk}{L^2}\right)^{-1/2} \qquad L = mv_0 s$$

$$= \pi - \pi\left(\frac{L^2}{L^2+mk}\right)^{1/2}$$

$$= \pi - \pi \frac{mv_0 s}{(m^2v_0^2s^2+mk)^{1/2}} = \pi - \pi \frac{s}{\left(s^2+\frac{k}{mv_0^2}\right)^{1/2}}$$

The parameter s can also be expressed in terms of the scattering angle, Θ, by using the expression {This is an alternative approach}

$$\Theta = |\pi - 2s\int_0^{u_0}\left[1 - s^2u^2 - V(u^{-1})/\tfrac{1}{2}mv_0^2\right]^{-\frac{1}{2}} du$$

where u_0 is the first positive root of denominator of the integrand.

$$V(r) = -\int_{\infty}^{r} F(r)\,dr \quad , \quad F(r) = \frac{k}{r^3}$$

$$= \frac{k}{2r^2}$$

$$V(u^{-1}) = \frac{k}{2}u^2$$

$$1 - s^2u_0^2 - \frac{k}{2}u_0^2/\tfrac{1}{2}mv_0^2 = 0$$

$$u_0^2\left(s^2+\frac{k}{mv_0^2}\right) = 1$$

$$u_0 = \left(s^2+\frac{k}{mv_0^2}\right)^{-1/2}$$

$$\Theta = \left| \pi - 2s \int_0^{u_0} \left[1 - u^2\left(s^2 + \frac{K}{mv_0^2}\right)\right]^{-1/2} du \right|$$

$$\text{let } u^2\left(s^2 + \frac{K}{mv_0^2}\right) = \sin^2\theta$$

$$u = \left(s^2 + \frac{K}{mv_0^2}\right)^{-1/2} \sin\theta$$

$$du = \left(s^2 + \frac{K}{mv_0^2}\right)^{-1/2} \cos\theta \, d\theta$$

$$= \left| \pi - 2s \int_{\theta=0}^{\theta=\theta_0} \frac{\left(s^2 + \frac{K}{mv_0^2}\right)^{-1/2} \cos\theta \, d\theta}{\left[1 - \sin^2\theta\right]^{1/2}} \right|$$

$$= \left| \pi - 2s\left(s^2 + \frac{K}{mv_0^2}\right)^{-1/2} \int_0^{\theta_0} d\theta \right|$$

$$= \left| \pi - 2s\left(s^2 + \frac{K}{mv_0^2}\right)^{-1/2} \sin^{-1}\left(u\left(s^2 + \frac{K}{mv_0^2}\right)^{1/2}\right) \Big|_0^{u_0 = \left(s^2 + \frac{K}{mv_0^2}\right)^{-1/2}} \right|$$

$$= \left| \pi - 2s\left(s^2 + \frac{K}{mv_0^2}\right)^{-1/2} \sin^{-1}(1) \right|$$

$$= \left| \pi - 2s\left(s^2 + \frac{K}{mv_0^2}\right)^{-1/2} \frac{\pi}{2} \right|$$

$$= \left| \pi - \pi s\left(s^2 + \frac{K}{mv_0^2}\right)^{-1/2} \right| \quad \{\text{agrees with first approach}\}$$

$$\Theta - \pi = -\pi s\left(s^2 + \frac{K}{mv_0^2}\right)^{-1/2}$$

$$\frac{\pi - \Theta}{\pi} = \frac{s}{\left(s^2 + \frac{K}{mv_0^2}\right)^{1/2}}$$

$$\frac{(\pi - \Theta)^2}{\pi^2} = \frac{s^2}{s^2 + \frac{K}{mv_0^2}}$$

$$\frac{(\pi - \Theta)^2}{\pi^2} = \frac{1}{1 + \frac{K}{mv_0^2 s^2}}$$

$$\frac{\pi^2}{(\pi - \Theta)^2} = 1 + \frac{K}{mv_0^2 s^2}$$

$$\frac{\pi^2}{(\pi-\Theta)^2} - 1 = \frac{K}{mv_0^2 s^2}$$

$$\frac{\pi^2-(\pi-\Theta)^2}{(\pi-\Theta)^2} = \frac{K}{mv_0^2 s^2}$$

$$s^2 = \frac{K}{mv_0^2}\,\frac{(\pi-\Theta)^2}{\pi^2-(\pi-\Theta)^2}$$

$$= \frac{K}{mv_0^2}\,\frac{(\pi-\Theta)^2}{\pi^2-(\pi^2-2\pi\Theta+\Theta^2)}$$

$$= \frac{K}{mv_0^2}\,\frac{(\pi-\Theta)^2}{\Theta(2\pi-\Theta)}$$

$$s = \left(\frac{K}{mv_0^2}\,\frac{(\pi-\Theta)^2}{\Theta(2\pi-\Theta)}\right)^{1/2}$$

Now,

$$\Theta = \pi - \pi\left(1+\frac{mk}{L^2}\right)^{-1/2} \qquad \text{where } L = mv_0 s$$

$$d\Theta = \frac{\pi}{2}\left(1+\frac{mk}{L^2}\right)^{-3/2}\left(\frac{-2mk}{L^3}\right)dL \qquad dL = mv_0\,ds$$

$$= \frac{-\pi mk}{L^3}\left(\frac{L^2+mk}{L^2}\right)^{-3/2} mv_0\,ds$$

$$= \frac{-\pi mk}{L^3}\left(\frac{L^2}{L^2+mk}\right)^{3/2} mv_0\,ds$$

$$= \frac{-\pi mk}{L^3}\,\frac{L^3}{(L^2+mk)^{3/2}}\,mv_0\,ds$$

$$= \frac{-\pi mk}{(L^2+mk)^{3/2}}\,mv_0\,ds$$

$$ds = \frac{(L^2+mk)^{3/2}}{\pi m^2 v_0 K}\,d\Theta \qquad L^2 = m^2v_0^2s^2$$

$$= m^2v_0^2\left(\frac{K}{mv_0^2}\,\frac{(\pi-\Theta)^2}{\Theta(2\pi-\Theta)}\right)$$

$$= mk\,\frac{(\pi-\Theta)^2}{\Theta(2\pi-\Theta)}$$

$$ds = \frac{\left(\frac{mk(\pi-\Theta)^2}{\Theta(2\pi-\Theta)} + mk\right)^{3/2}}{\pi m^2 v_0 K}\, d\Theta$$

$$= \frac{(mk)^{3/2}\left[\frac{(\pi-\Theta)^2}{\Theta(2\pi-\Theta)} + 1\right]^{3/2}}{\pi m^2 v_0 K}\, d\Theta$$

$$= \frac{(mk)^{3/2}}{\pi m^2 v_0 K}\left[\frac{(\pi-\Theta)^2 + \Theta(2\pi-\Theta)}{\Theta(2\pi-\Theta)}\right]^{3/2} d\Theta$$

$$= \frac{(mk)^{3/2}}{\pi m^2 v_0 K}\left[\frac{\pi^2 - 2\pi\Theta + \Theta^2 + 2\pi\Theta - \Theta^2}{\Theta(2\pi-\Theta)}\right]^{3/2} d\Theta$$

$$ds = \frac{(mk)^{3/2}}{\pi m^2 v_0 K}\left[\frac{\pi^2}{\Theta(2\pi-\Theta)}\right]^{3/2} d\Theta$$

Now,

$$d\sigma = 2\pi s\, ds$$

$$= 2\pi\left[\frac{K}{mv_0^2}\,\frac{(\pi-\Theta)^2}{\Theta(2\pi-\Theta)}\right]^{1/2}\frac{(mk)^{3/2}}{\pi m^2 v_0 K}\,\frac{\pi^3}{[\Theta(2\pi-\Theta)]^{3/2}}\, d\Theta$$

$$= 2\pi^3\left(\frac{K}{mv_0^2}\right)^{1/2}\frac{(mk)^{3/2}}{m^2 v_0 K}\,\frac{(\pi-\Theta)}{\Theta^2(2\pi-\Theta)^2}\, d\Theta$$

$$\boxed{d\sigma = 2\pi^3\,\frac{K}{mv_0^2}\,\frac{(\pi-\Theta)}{\Theta^2(2\pi-\Theta)^2}\, d\Theta}$$

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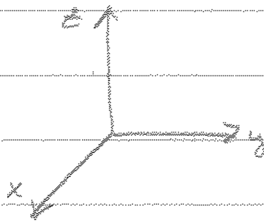


$$\vec{B} = B_0 \hat{z}$$

$$\vec{E} = E_0 \hat{x} \sin\left(\frac{qB_0}{mc} t\right)$$

$$\vec{F} = q\vec{E} + \frac{q}{c} \vec{v} \times \vec{B}$$

$$F_x = qE_0 \sin\left(\frac{qB_0}{mc} t\right) + \frac{q}{c} v_y B_0$$

$$F_y = -\frac{q}{c} v_x B_0$$

$$F_z = 0 \rightarrow \boxed{z = 0}$$

$$m\ddot{x} = qE_0 \sin\left(\frac{qB_0}{mc} t\right) + \frac{qB_0}{c} \dot{y}$$

$$\ddot{x} = \frac{q}{m} E_0 \sin\left(\frac{qB_0}{mc} t\right) + \frac{qB_0}{mc} \dot{y}$$

$$m\ddot{y} = -\frac{qB_0}{c} \dot{x} \qquad \vec{r}(0) = 0$$

$$\ddot{y} = -\frac{qB_0}{mc} \dot{x} \qquad \vec{v}(0) = 0, \ t(0) = 0$$

$$\int_0^{v_y} dv = \int_0^x -\frac{qB_0}{mc} dx$$

$$\dot{y} = -\frac{qB_0}{mc} x$$

$$\ddot{x} = \frac{q}{m} E_0 \sin\left(\frac{qB_0}{mc} t\right) - \left(\frac{qB_0}{mc}\right)^2 x$$

$$\ddot{x} + \left(\frac{qB_0}{mc}\right)^2 x = \frac{q}{m} E_0 \sin\left(\frac{qB_0}{mc} t\right)$$

Homogeneous solution

$$\ddot{x} + \left(\frac{qB_0}{mc}\right)^2 x = 0 \qquad x = e^{\lambda t}$$

$$\lambda^2 + \left(\frac{qB_0}{mc}\right)^2 = 0$$

$$\lambda^2 = -\left(\frac{qB_0}{mc}\right)^2$$

$$\lambda = \pm i \frac{qB_0}{mc}$$

$$x = A\cos\left(\frac{qB_0}{mc} t\right) + B\sin\left(\frac{qB_0}{mc} t\right)$$

$$x_h = A\cos(\omega t) + B\sin(\omega t), \ \omega = \frac{qB_0}{mc}$$

Particular solution

$$x_p = At\cos(\omega t) + Bt\sin(\omega t)$$

$$\dot{x} = -At\omega\sin\omega t + A\cos\omega t + Bt\omega\cos\omega t + B\sin\omega t$$

$$\ddot{x} = -At\omega^2\cos\omega t - A\omega\sin kt - Ak\sin\omega t - Bt\omega^2\sin\omega t + Bk\cos\omega t$$
$$+ B\omega\cos\omega t$$

$$= (-At\omega^2 + 2B\omega)\cos\omega t - (Bt\omega^2 + 2A\omega)\sin\omega t$$

Substituting $\ddot{x}$ and x into the differential equation gives

$$(-At\omega^2 + 2B\omega)\cos\omega t - (Bt\omega^2 + 2A\omega)\sin\omega t +$$
$$\omega^2[At\cos\omega t + Bt\sin\omega t] = \frac{q}{m}E_0\sin\omega t$$

Equating coefficients,

$$-At\omega^2 + 2B\omega + At\omega^2 = 0 \rightarrow B = 0$$

$$-Bt\omega^2 - 2A\omega + Bt\omega^2 = \frac{q}{m}E_0$$

$$A = -\frac{qE_0}{2m\omega}$$

$$x_p = \frac{-qE_0}{2m\omega}t\cos(\omega t)$$

General solution is

$$x = x_h + x_p$$

$$x = A\cos\omega t + B\sin\omega t - \frac{qE_0}{2m\omega}t\cos\omega t$$

$$0 = x(0) = A$$

$$x = B\sin\omega t - \frac{qE_0}{2m\omega}t\cos\omega t$$

$$\dot{x} = B\omega\cos\omega t + \frac{qE_0}{2m}t\sin\omega t - \frac{qE_0}{2m\omega}\cos\omega t$$

$$0 = v(0) = B\omega - \frac{qE_0}{2m\omega}$$

$$B = \frac{qE_0}{2m\omega^2}$$

$$x = \frac{qE_0}{2m\omega^2}\sin\omega t - \frac{qE_0}{2m\omega}t\cos\omega t$$

$$x = \frac{qE_0}{2m\omega^2}(\sin\omega t - \omega t\cos\omega t)$$

$$\dot{y} = -\omega x$$

$$\int_0^y dy = -\omega \int_0^t x\,dt$$

$$y = -\omega \int_0^t \left[\frac{qE_0}{2m\omega^2} \sin\omega t - \frac{qE_0}{2m\omega} t\cos\omega t \right] dt$$

$$= \frac{-qE_0}{2m\omega} \int_0^t \left[\sin\omega t - \omega t\cos\omega t \right] dt$$

$$= \frac{-qE_0}{2m\omega} \left[\int_0^t \sin\omega t\,dt - \omega \int_0^t t\cos\omega t\,dt \right]$$

let $u = t$ $du = dt$

$dv = \cos\omega t$ $v = \frac{1}{\omega}\sin\omega t$

$$= \frac{-qE_0}{2m\omega} \left[-\frac{1}{\omega}\cos\omega t \Big|_0^t - \omega \left(\frac{t}{\omega}\sin\omega t \Big|_0^t - \int_0^t \frac{1}{\omega}\sin\omega t\,dt \right) \right]$$

$$= \frac{-qE_0}{2m\omega} \left[-\frac{1}{\omega}\cos\omega t \Big|_0^t - t\sin\omega t \Big|_0^t - \frac{1}{\omega}\cos\omega t \Big|_0^t \right]$$

$$= \frac{-qE_0}{2m\omega} \left[-\frac{1}{\omega}\cos\omega t + \frac{1}{\omega} - t\sin\omega t - \frac{1}{\omega}\cos\omega t + \frac{1}{\omega} \right]$$

$$= \frac{-qE_0}{2m\omega} \left[-\frac{2}{\omega}\cos\omega t + \frac{2}{\omega} - t\sin\omega t \right]$$

$$y = \frac{qE_0}{2m\omega^2} \left[2\cos\omega t - 2 + \omega t\sin\omega t \right]$$

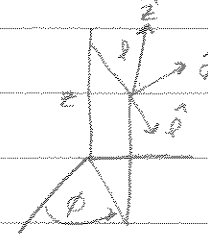
$$\omega = \frac{qB_0}{mc}$$

$$\boxed{\begin{aligned} x &= \frac{E_0 mc^2}{2B_0^2 q} \left[\sin\left(\frac{qB_0}{mc}t\right) - \frac{qB_0}{mc} t\cos\left(\frac{qB_0}{mc}t\right) \right] \\ y &= \frac{E_0 mc^2}{2B_0^2 q} \left[2\cos\left(\frac{qB_0}{mc}t\right) - 2 + \frac{qB_0}{mc} t\sin\left(\frac{qB_0}{mc}t\right) \right] \end{aligned}}$$

3-75

$\vec{B} = B\hat{z}$

$E = \frac{a}{\rho}\hat{\rho}$



$\frac{d\hat{\rho}}{d\phi} = \hat{\phi} \qquad \frac{d\hat{\phi}}{d\phi} = -\hat{\rho}$

a) $\vec{F} = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B}$

$$\frac{q}{c}\vec{v} \times \vec{B} = \frac{q}{c}\begin{vmatrix} \hat{\rho} & \hat{\phi} & \hat{z} \\ \dot{\rho} & \rho\dot{\phi} & \dot{z} \\ 0 & 0 & B \end{vmatrix} \qquad v_\phi = \rho\dot{\phi}$$

$$= \frac{q}{c}\rho\dot{\phi}B\hat{\rho} - \frac{q}{c}\dot{\rho}B\hat{\phi}$$

$$\vec{F} = q\frac{a}{\rho}\hat{\rho} + \frac{qB}{c}\rho\dot{\phi}\hat{\rho} - \frac{qB}{c}\dot{\rho}\hat{\phi}$$

For cylindrical coordinates

$$\vec{a} = (\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} + (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi} + \ddot{z}\hat{z}$$

$$\vec{F} = m\vec{a} = q\frac{a}{\rho}\hat{\rho} + \frac{qB}{c}\rho\dot{\phi}\hat{\rho} - \frac{qB}{c}\dot{\rho}\hat{\phi}$$

$$\boxed{(\ddot{\rho} - \rho\dot{\phi}^2)\hat{\rho} = \frac{qa}{m\rho}\hat{\rho} + \frac{qB}{mc}\rho\dot{\phi}\hat{\rho}, \quad (\rho\ddot{\phi} + 2\dot{\rho}\dot{\phi})\hat{\phi} = -\frac{qB}{mc}\dot{\rho}\hat{\phi}}$$

$$\rho^2\ddot{\phi} + 2\rho\dot{\rho}\dot{\phi} = -\frac{qB}{mc}\dot{\rho}\rho$$

$\ddot{z} = 0$ $\qquad m\rho^2\ddot{\phi} + 2m\rho\dot{\rho}\dot{\phi} = \frac{qB}{c}\dot{\rho}\rho$

$\dot{z} = \dot{z}_0$ $\qquad \frac{d}{dt}(m\rho^2\dot{\phi}) = \frac{qB}{c}\dot{\rho}\rho$

$\boxed{z = z_0 + \dot{z}_0 t}$ $\qquad \frac{dL}{dt} = \frac{qB}{c}\dot{\rho}\rho$

$$dL = \frac{qB}{c}\rho\, d\rho$$

$$L_B = \frac{qB}{2c}\rho^2$$

b) L_B is the angular momentum due to the B field. Another component of L, due to the central force $q\vec{E}$ acting on a $\hat{\phi}$ component is $L_\phi = m\rho^2\dot{\phi}$. Therefore, the total angular momentum is a constant.

$$L_\phi + L_B = K$$

$$\boxed{m\rho^2\dot{\phi} + \frac{qB}{2c}\rho^2 = K}$$

c) $T + qU = E$ $\qquad U = -\int_{\rho_s}^{\rho} \vec{E}\cdot d\vec{r}$

$\frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}mv_{Tan}^2 + qU = E$ $\qquad v_{Tan} = \rho\dot{\phi}$

$\frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\dot{\phi}^2 + qU = E$ $\qquad \dot{\phi} = \dfrac{K - \frac{qB}{2c}\rho^2}{m\rho^2}$

$\frac{1}{2}m\dot{\rho}^2 + \frac{1}{2}m\rho^2\left(\frac{2Kc - qB\rho^2}{2mc\rho^2}\right)^2 + qU = E$ $\qquad = \dfrac{K}{m\rho^2} - \dfrac{qB}{2mc} = \dfrac{2Kc - qB\rho^2}{2mc\rho^2}$

$$\frac{1}{2}m\dot{\rho}^2 + \frac{(2Kc - qB\rho^2)^2}{8mc^2\rho^2} + qU = E$$

$$\frac{1}{2}m\dot{\rho}^2 = E - qU - \frac{(2Kc - qB\rho^2)^2}{8mc^2\rho^2}$$

$$\frac{d\rho}{dt} = \dot{\rho} = \sqrt{\frac{2}{m}}\left(E - qU - \frac{(2Kc - qB\rho^2)^2}{8mc^2\rho^2}\right)^{1/2}$$

$$\int_{\rho_0}^{\rho} \frac{d\rho}{\left(E - qU - \frac{(2Kc - qB\rho^2)^2}{8mc^2\rho^2}\right)^{1/2}} = \sqrt{\frac{2}{m}}\,t$$

$U = -\int_{\rho_s}^{\rho} \frac{a}{\rho}\,d\rho$

$= -a\ln\rho\Big|_{\rho_s}^{\rho}$

$= -a\ln\frac{\rho}{\rho_s}$

$= a\ln\frac{\rho_s}{\rho}, \quad 0 < \rho \le \rho_s$

$$V'(\rho) = qU + \frac{(2Kc - qB\rho^2)^2}{8mc^2\rho^2}$$

$$= qa\ln\frac{\rho_s}{\rho} + \frac{(2Kc - qB\rho^2)^2}{8mc^2\rho^2}$$

$$= qa\ln\frac{\rho_s}{\rho} + \frac{4K^2c^2 - 4KcqB\rho^2 + q^2B^2\rho^4}{8mc^2\rho^2}$$

$$V'(\rho) = qa\ln\frac{\rho_s}{\rho} + \frac{K^2}{2m\rho^2} - \frac{KqB}{2mc} + \frac{q^2B^2\rho^2}{8mc^2}$$

$$V'(\rho) = qU + \frac{K^2}{2m}\frac{1}{\rho^2} - \frac{KqB}{2mc} + \frac{q^2B^2}{8mc^2}\rho^2$$

$$\frac{dV'}{d\rho} = -qE - \frac{K^2}{m}\frac{1}{\rho^3} + \frac{q^2B^2}{4mc^2}\rho$$

$$0 = -qE - \frac{K^2}{m}\frac{1}{\rho^3} + \frac{q^2B^2}{4mc^2}\rho$$

$$= -qE\rho^3 - \frac{K^2}{m} + \frac{q^2B^2}{4mc^2}\rho^4$$

$$= -aq\rho^2 - \frac{K^2}{m} + \frac{q^2B^2}{4mc^2}\rho^4$$

$$\rho^4 - \frac{4mc^2}{q^2B^2}aq\rho^2 - \frac{4mc^2}{q^2B^2}\frac{K^2}{m} = 0$$

$$\rho^4 - \frac{4mc^2a}{B^2q}\rho^2 - \frac{4K^2c^2}{q^2B^2} = 0$$

$$\rho^4 - \frac{4mc^2a}{B^2q}\rho^2 + \left(\frac{2mc^2a}{B^2q}\right)^2 = \left(\frac{2Kc}{qB}\right)^2 + \left(\frac{2mc^2a}{B^2q}\right)^2$$

$$\left(\rho^2 - \frac{2mc^2a}{B^2q}\right)^2 = \frac{4K^2c^2}{q^2B^2} + \frac{4m^2c^4a^2}{B^4q^2}$$

$$= \frac{4K^2B^2c^2 + 4m^2c^4a^2}{B^4q^2}$$

$$\rho_0^2 = \frac{2mc^2a}{B^2q} \pm \frac{2c}{B^2q}\left(K^2B^2 + m^2c^2a^2\right)^{1/2}$$

d) For circular motion $\dot{z}_0 = 0$, $\dot{\rho}_0 = 0$

$$(\ddot{\rho} - \rho\dot{\phi}^2) = \frac{qa}{m\rho} + \frac{qB}{mc}\rho\dot{\phi}$$

$$-\rho\dot{\phi}^2 = \frac{qa}{m\rho} + \frac{qB}{mc}\rho\dot{\phi}$$

$$\dot{\phi}^2 + \frac{qB}{mc}\dot{\phi} + \frac{qa}{m\rho^2} = 0$$

$$\left(\dot{\phi}_0 + \frac{qB}{2mc}\right)^2 = \left(\frac{qB}{2mc}\right)^2 - \frac{qa}{m\rho_0^2} \quad \{\rho = \rho_0\}\{\dot{\phi} = \dot{\phi}_0\}$$

$$\boxed{\dot{\phi}_0 = -\frac{qB}{2mc} \pm \left(\left(\frac{qB}{2mc}\right)^2 - \frac{qa}{m\rho_0^2}\right)^{1/2}}$$

e) $$\omega_\rho^2 = \frac{1}{m}\left(\frac{d^2V}{d\rho^2}\right)_{\rho_0}$$

$$= \frac{1}{m}\frac{d}{d\rho}\left[-\frac{qa}{\rho} - \frac{K^2}{m}\frac{1}{\rho^3} + \frac{q^2B^2}{4mc^2}\rho\right]_{\rho_0}$$

$$= \frac{1}{m}\left[\frac{qa}{\rho^2} + \frac{3K^2}{m\rho^4} + \frac{q^2B^2}{4mc^2}\right]_{\rho_0}$$

$$= \left(\frac{qa}{m\rho^2} + \frac{3K^2}{m^2\rho^4} + \left(\frac{qB}{2mc}\right)^2\right)_{\rho_0} \qquad K = \frac{2mc\rho^2\dot{\phi} + qB\rho^2}{2c}$$

$$= \rho^2\left[\frac{2mc\dot{\phi} + qB}{2c}\right]$$

$$= \left(\frac{qa}{m\rho^2} + \frac{3\left[2mc\dot{\phi} + qB\right]^2}{4m^2c^2} + \left(\frac{qB}{2mc}\right)^2\right)_{\rho_0} \qquad K^2 = \rho^4\frac{\left[2mc\dot{\phi} + qB\right]^2}{4c^2}$$

$$= \left(\frac{qa}{m\rho^2} + 3\left(\frac{2mc\dot{\phi} + qB}{2mc}\right)^2 + \left(\frac{qB}{2mc}\right)^2\right)_{\rho_0}$$

$$= \left(\frac{qa}{m\rho^2} + 3\frac{\left(4m^2c^2\dot{\phi}^2 + 4mcqB\dot{\phi} + (qB)^2\right)}{(2mc)^2} + \left(\frac{qB}{2mc}\right)^2\right)_{\rho_0}$$

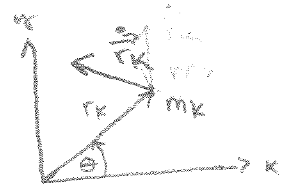
$$= \left(\frac{qa}{m\rho^2} + \frac{3m^2c^2\left(-\frac{qa}{m\rho^2} - \frac{qB}{mc}\dot{\phi}\right)}{m^2c^2} + \frac{3mcqB\dot{\phi}}{m^2c^2} + 4\left(\frac{qB}{2mc}\right)^2\right)_{\rho_0}$$

$$= \left(\frac{qa}{m\rho^2} - \frac{3qa}{m\rho^2} - \frac{3qB\dot{\phi}}{mc} + \frac{3qB\dot{\phi}}{mc} + 4\left(\frac{qB}{2mc}\right)^2\right)_{\rho_0}$$

$$= 4\left(\frac{qB}{2mc}\right)^2 - \frac{2qa}{m\rho_0^2}$$

$$\boxed{\omega_\rho = \pm 2\left[\left(\frac{qB}{2mc}\right)^2 - \frac{qa}{2m\rho_0^2}\right]^{1/2}}$$

4-1



$$\vec{L}_{Ko} = m_K \vec{r}_K \times \dot{\vec{r}}_K$$

$$\dot{\vec{r}}_K = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\vec{L}_{Ko} = m_K \vec{r}_K \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$= m_K r\hat{r} \times (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$$

$$= m_K [r\hat{r} \times \dot{r}\hat{r} + r\hat{r} \times r\dot{\theta}\hat{\theta}]$$

$$\vec{L}_{Ko} = m_K r\hat{r} \times r\dot{\theta}\hat{\theta} = mr^2\dot{\theta}\hat{z}$$

$$\frac{d\vec{L}_{Ko}}{dt} = \frac{d}{dt}(m_K \vec{r}_K \times \dot{\vec{r}}_K)$$

$$= \vec{r}_K \times \frac{d}{dt}(m\dot{\vec{r}}) + \frac{d\vec{r}}{dt} \times m\dot{\vec{r}}$$

$$= \vec{r}_K \times m\ddot{\vec{r}} + \dot{\vec{r}} \times m\dot{\vec{r}}$$

$$= \vec{r}_K \times m\ddot{\vec{r}}$$

$$= \vec{r}_K \times \vec{F} \quad \{\vec{F} = m\ddot{\vec{r}}\}$$

If the external applied force is zero then

$$\frac{d\vec{L}_{Ko}}{dt} = 0$$

$$\vec{L}_{Ko} = \text{constant}$$

$$\vec{L}_{Ko} = mr^2\dot{\theta}\hat{z} = \text{constant}$$

4-29 A spaceship of mass m, initial velocity $\vec{v}_0$ approaches the moon and passes by. The distance of closest approach is R (measured from the moon's center). The velocity $\vec{v}_0$ is perpendicular to the orbit velocity $\vec{V}$ of the moon. Show that if the spaceship passes behind the moon, it will gain kinetic energy and calculate the increase of kinetic energy as it leaves the vicinity of the moon. Assume $M \gg m$, where M is the moon's mass.

This is a 2-body problem which can be seperated into two one body systems. the one system is the center of mass and the other is the motion of one mass relative to the other. A derivation of this approach is instructive.

The equations of motion in a reference frame with an arbitrary origin are

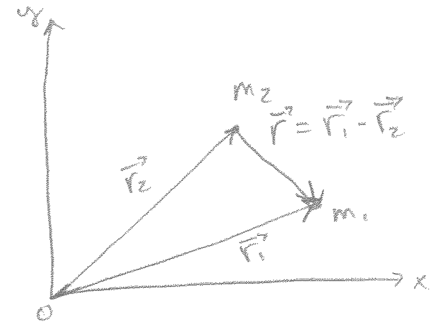
$$m_1\ddot{\vec{r}}_1 = -\frac{dV}{dr_1}\hat{r}_1$$

$$m_2\ddot{\vec{r}} = -\frac{dV}{dr_2}\hat{r}_2$$

$$V = \frac{-Gm_1m_2}{|\vec{r}_1 - \vec{r}_2|}$$

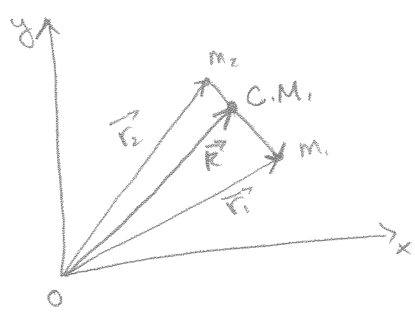
Let $\vec{r} = \vec{r}_1 - \vec{r}_2$

$$V = \frac{-Gm_1m_2}{r}$$



Now, the center of mass (C.M.) is on the line between m_1 and m_2. Its location is found from

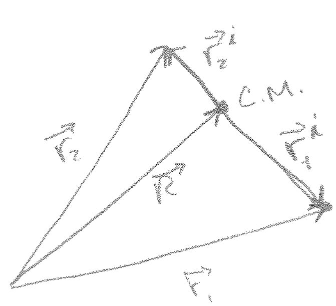
$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2}$$



the total momentum of m_1 and m_2 is equivalent to $(m_1+m_2)\dot{\vec{R}}$

Let $r_1' = \vec{r}_1 - \vec{R}$

$r_2' = \vec{r}_2 - \vec{R}$



$$\vec{v}_1' = \vec{v}_1 - \vec{V}_{cm} \qquad \vec{v}_2' = \vec{v}_2 - \vec{V}_{cm}$$

$$\vec{v}_1 = \vec{v}_1' + \vec{V}_{cm} \qquad \vec{v}_2 = \vec{v}_2' + \vec{V}_{cm}$$

where $\vec{v}_1$ and $\vec{v}_2$ are in the laboratory frame, and $\vec{v}_1'$ and v_2' are in the center of mass frame. The relation between the center of mass coordinates and the relative coordinate $\vec{r}$ is found by:

$$\vec{R} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1+m_2}, \quad \vec{r} = \vec{r}_1 - \vec{r}_2$$

$$(m_1+m_2)\vec{R} = m_1\vec{r}_1 + m_2(\vec{r}_1 - \vec{r})$$
$$= (m_1+m_2)\vec{r}_1 - m_2\vec{r}$$
$$\vec{r}_1 = \vec{R} + \frac{m_2}{m_1+m_2}\vec{r}$$

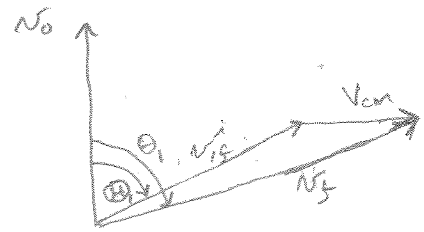
$$(m_1+m_2)\vec{R} = m_1(\vec{r}+\vec{r}_2) + m_2\vec{r}_2$$
$$= (m_1+m_2)\vec{r}_2 + m_1\vec{r}$$
$$\vec{r}_2 = \vec{R} - \frac{m_1}{m_1+m_2}\vec{r}$$

Therefore,

$$\vec{r}_1' = \frac{m_2}{m_1+m_2}\vec{r} \qquad \vec{r}_2' = -\frac{m_1}{m_1+m_2}\vec{r}$$

$$\vec{v}_1' = \frac{m_2}{m_1+m_2}\vec{v} = \frac{\mu}{m_1}\vec{v} \qquad \vec{v}_2' = -\frac{m_1}{m_1+m_2}\vec{v} = -\frac{\mu}{m_2}\vec{v}, \quad \mu = \frac{m_1 m_2}{m_1+m_2}$$

where $\vec{v}$ is the relative velocity between the two masses.



Θ_1 = scatter angle in C.M. frame

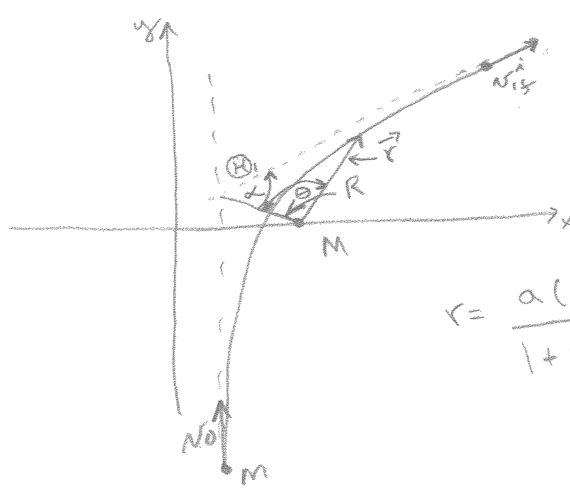
θ_1 = scatter angle in lab frame

$$\vec{V}_{cm} = \frac{MV\hat{x} + mv_0\hat{y}}{M+m}$$

$$\doteq \frac{MV\hat{x} + mv_0\hat{y}}{M} \quad \{M >> m\}$$

$$\doteq V\hat{x} + \frac{m}{M}v_0\hat{y}$$

$$\doteq V\hat{x}$$



$$r = \frac{a(\epsilon^2-1)}{1+\epsilon\cos\theta}, \text{ at } r=\infty \quad 1+\epsilon\cos\theta = 0 \quad \{\theta=\alpha\}$$

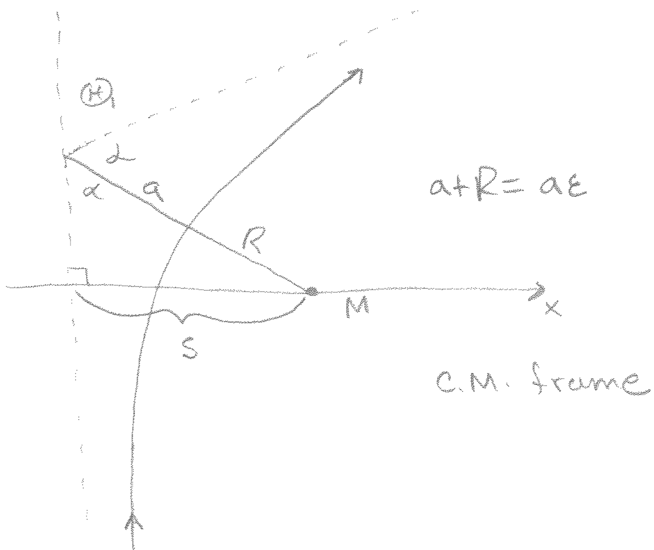
$$\cos\alpha = \frac{\pm 1}{\epsilon}$$

$$\cos\alpha = \frac{1}{\epsilon} \{+\text{ branch}\}$$

The scattering angle, Θ_1, is related to α by:

$$\Theta_1 = \pi - 2\alpha$$

Therefore, α must be determined in terms of the given parameters.



For an ellipse or hyperbola, the semi-major axis, a, is

$$a = \left|\frac{K}{2E}\right| = \frac{GmM}{2E}$$

where E is the energy in the C.M. frame and constant $\rightarrow E = \frac{1}{2}\mu v_0^2$

The minimum radius for the hyperbola is

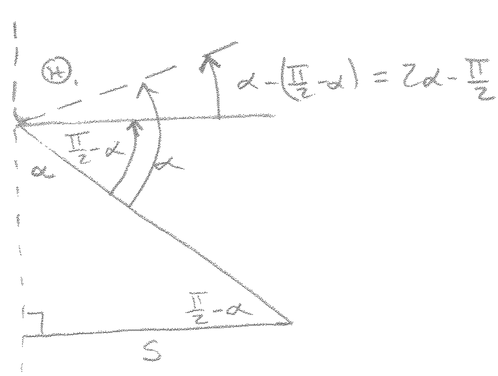
$$r_{min} = a(\varepsilon - 1)$$

$$R = r_{min}$$

$$R = \frac{K}{2E}(\varepsilon - 1)$$

$$\cos\alpha = \frac{1}{\varepsilon}$$

$$R = \frac{K}{\mu v_0^2}\left(\frac{1}{\cos\alpha} - 1\right)$$



$$\frac{R\mu v_0^2}{K} + 1 = \frac{1}{\cos\alpha}$$

$$\frac{R\mu v_0^2 + K}{K} = \frac{1}{\cos\alpha}$$

$$\cos\alpha = \frac{K}{R\mu v_0^2 + K}$$

$$\cos\alpha = \frac{K}{R\mu v_0^2 + K} \qquad K = GmM, \quad \mu = \frac{mM}{m+M}$$

$$= \frac{GmM}{R\frac{mM}{m+M}v_0^2 + GmM}$$

$$\doteq \frac{GmM}{Rmv_0^2 + GmM} \quad \{M >> m\}$$

$$\doteq \frac{GM}{Rv_0^2 + GM}$$

$$\doteq = \frac{\frac{GM}{R}}{v_0^2 + \frac{GM}{R}}$$

$$\cos\alpha \doteq \frac{C}{v_0^2 + C} \quad \{C = \frac{GM}{R}\}$$

$$\sin^2\alpha = 1 - \frac{C^2}{(v_0^2+C)^2}$$

$$= \frac{(v_0^2+C)^2 - C^2}{(v_0^2+C)^2}$$

$$= \frac{v_0^4 + 2v_0^2C}{(v_0^2+C)^2}$$

$$\sin\alpha = \frac{(v_0^4 + 2v_0^2C)^{1/2}}{(v_0^2+C)} = \frac{v_0(v_0^2+2C)^{1/2}}{(v_0^2+C)}$$

the components of v_f' are in the lab frame

$$\vec{v}_f' = v_f'\cos(2\alpha - \tfrac{\pi}{2})\hat{x} + v_f'\sin(2\alpha - \tfrac{\pi}{2})\hat{y}$$

$$= v_f'(\sin 2\alpha\,\hat{x} - \cos 2\alpha\,\hat{y})$$

$$= v_f'(2\sin\alpha\cos\alpha\,\hat{x} - (2\cos^2\alpha - 1)\hat{y})$$

$$= \frac{\mu}{m}v(2\sin\alpha\cos\alpha\,\hat{x} - (2\cos^2\alpha - 1)\hat{y})$$

$$= \frac{M}{m+M}v(2\sin\alpha\cos\alpha\,\hat{x} - (2\cos^2\alpha - 1)\hat{y})$$

$$v_f' \doteq v(2\sin\alpha\cos\alpha\,\hat{x} - (2\cos^2\alpha - 1)\hat{y}) \quad \{M >> m\}$$

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\sin 2\alpha = 2\sin\alpha\cos\alpha$$
$$\cos 2\alpha = 2\cos^2\alpha - 1$$

At $\theta = \alpha$ (along the asymptote), $v = v_0$.

$$\vec{v}_f = \vec{v}_f^{\,i} + \vec{V}_{cm} \quad \{\text{laboratory frame}\}$$

$$= (v_0 2\sin\alpha\cos\alpha + V)\hat{x} - v_0(2\cos^2\alpha - 1)\hat{y}$$

$$v_f^2 = \vec{v}_f \cdot \vec{v}_f$$

$$= (2v_0\sin\alpha\cos\alpha + V)^2 + (v_0(2\cos^2\alpha - 1))^2$$

$$= 4v_0^2\sin^2\alpha\cos^2\alpha + 4Vv_0\sin\alpha\cos\alpha + v_0^2(4\cos^4\alpha - 4\cos^2\alpha + 1) + V^2$$

$$= 4v_0^2\sin^2\alpha\cos^2\alpha + 4Vv_0\sin\alpha\cos\alpha + 4v_0^2\cos^4\alpha - 4v_0^2\cos^2\alpha + v_0^2 + V^2$$

$$= 4Vv_0\sin\alpha\cos\alpha + 4v_0^2\sin^2\alpha\cos^2\alpha + 4v_0^2\cos^4\alpha - 4v_0^2\cos^2\alpha + v_0^2 + V^2$$

$$= 4Vv_0\sin\alpha\cos\alpha + 4v_0^2\cos^2\alpha(\sin^2\alpha + \cos^2\alpha - 1) + v_0^2 + V^2$$

$$v_f^2 = 4Vv_0\sin\alpha\cos\alpha + v_0^2 + V^2$$

The final kinetic energy of m is

$$E_f = \frac{1}{2}m(4Vv_0\sin\alpha\cos\alpha + v_0^2 + V^2)$$

The intial kinetic energy of m is

$$E_0 = \frac{1}{2}mv_0^2$$

The change in kinetic energy is

$$\Delta E = E_f - E_0$$

$$\Delta E = \frac{1}{2}m(4Vv_0\sin\alpha\cos\alpha + V^2)$$

$$\sin\alpha\cos\alpha = \frac{v_0(v_0^2 + 2C)^{1/2}}{v_0^2 + C} \cdot \frac{C}{v_0^2 + C}$$

$$\Delta E = \frac{1}{2}m\left(\frac{4Vv_0^2 C(v_0^2 + 2C)^{1/2}}{(v_0^2 + C)^2} + V^2\right)$$

$$\Delta E = \frac{1}{2}m\left[\frac{\frac{4GM}{R}Vv_0^2\left(v_0^2 + \frac{2GM}{R}\right)^{1/2}}{\left(v_0^2 + \frac{GM}{R}\right)^2} + V^2\right]$$

4-31 Show that if the incident particle is much heavier than the target particle ($m_1 >> m_2$), the Rutherford scattering cross section in laboratory coordinates is approximately

$$d\sigma \doteq \left(\frac{q_1 q_2}{2 m_2 v_0^2}\right)^2 \frac{4\gamma^2}{\left[1-(1-\gamma^2\theta_1^2)^{1/2}\right]^2 (1-\gamma^2\theta_1^2)^{1/2}} 2\pi \sin\theta_1 \, d\theta_1$$

if $\gamma\theta_1 < 1$, where $\gamma = \frac{m_1}{m_2}$. Otherwise, $d\sigma = 0$

---

the relation between θ_1 and the relative (C.M.) frame Θ is

$$\tan\theta_1 = \frac{\sin\Theta}{\cos\Theta + \frac{m_1}{m_2}} = \frac{\sin\Theta}{\cos\Theta + \gamma}$$

The scattering formula is

$$d\sigma = \left(\frac{q_1 q_2}{2 m v_0^2}\right)^2 \frac{2\pi \sin\Theta}{\sin^4\left(\frac{\Theta}{2}\right)} d\Theta$$

If $\gamma\theta_1 < 1$ and $\gamma >> 1$ $\{m_1 >> m_2\}$ then $\theta_1 << 1$

$$\tan\theta_1 \doteq \theta_1$$

$$\theta_1 \doteq \frac{\sin\Theta}{\cos\Theta + \gamma}$$

$$\theta_1 \doteq \frac{\sin\Theta}{\gamma} \quad \{\gamma >> \cos\Theta\}$$

$$\gamma\theta_1 \doteq \sin\Theta$$

Now,

$$\sin^4\left(\frac{\Theta}{2}\right) = \left(\frac{1-\cos\Theta}{2}\right)^2 = \frac{\left[1-(1-\gamma^2\theta_1^2)^{1/2}\right]^2}{4}, \quad \cos\Theta = \left(1 - \sin^2\Theta\right)^{1/2} = \left(1-\gamma^2\theta_1^2\right)^{1/2}$$

Also,

$$\theta_1 \doteq \frac{\sin\Theta}{\gamma} \rightarrow \sin\theta_1 = \frac{\sin\Theta}{\gamma} \rightarrow \gamma\sin\theta_1 = \sin\Theta$$

Taking the derivative of $\gamma\theta_1 = \sin\Theta$ gives

$$\gamma d\theta_1 = \cos\Theta \, d\Theta$$

$$d\Theta = \frac{\gamma d\theta_1}{\cos\Theta}$$

$$d\Theta = \frac{\gamma d\theta_1}{(1-\gamma^2\theta_1^2)^{1/2}}$$

Finally, substituting into the scattering formula:

$$d\sigma \doteq \left(\frac{q_1 q_2}{2mv_0^2}\right)^2 \frac{2\pi\gamma\sin\theta_1}{\frac{[1-(1-\gamma^2\theta_1^2)^{1/2}]^2}{4}} \frac{\gamma d\theta_1}{(1-\gamma^2\theta_1^2)^{1/2}}$$

$$d\sigma \doteq \left(\frac{q_1 q_2}{2mv_0^2}\right)^2 \frac{4\gamma^2}{[1-(1-\gamma^2\theta_1^2)^{1/2}]^2(1-\gamma^2\theta_1^2)^{1/2}} 2\pi\sin\theta_1 d\theta_1$$

when $\gamma\theta_1 \geq 1$, $d\sigma$ becomes imaginary $\{\gamma\theta_1 > 1\}$ or undefined $\{\gamma\theta_1 = 1\}$. So, $d\sigma = 0$ otherwise. Physically, this implies θ_1 is large but a massive incident particle will always have θ_1 small, so only γ can increase and the scattering angle θ_1 tends to zero.

4-33 Assume that $m_2 >> m_1$, and that $\Theta = \theta_1 + \delta$, in Eq. (4.117). Find a formula for δ in terms of θ_1. Show that the first order correction to the Rutherford scattering cross section [Eq. (3.276)], due to the finite mass of m_2, vanishes.

Eq. (4.117)

$$\tan\theta_1 = \frac{\sin\Theta}{\cos\Theta + \frac{m_1}{m_2}}$$

$$\tan\theta_1 = \frac{\sin(\theta_1+\delta)}{\cos(\theta_1+\delta)+\gamma} \qquad \gamma = \frac{m_1}{m_2}$$

$$\{\cos(\theta_1+\delta)+\gamma\}\tan\theta_1 = \sin(\theta_1+\delta)$$

$\sin(A+B) = \sin A\cos B + \cos B \sin A$

$\cos(A+B) = \cos A\cos B - \sin A \sin B$

$$\{\cos\theta_1\cos\delta - \sin\theta_1\sin\delta + \gamma\}\tan\theta_1 = \sin\theta_1\cos\delta + \cos\theta_1\sin\delta$$

$$\sin\theta_1\cos\delta - \frac{\sin^2\theta_1}{\cos\theta_1}\sin\delta + \gamma\frac{\sin\theta_1}{\cos\theta_1} = \sin\theta_1\cos\delta + \cos\theta_1\sin\delta$$

when $m_2 >> m_1$, $\theta_1 \doteq \Theta$. Therefore, $\delta << 1$.

$$\cos\alpha = 1 - \frac{\alpha^2}{2} + \cdots$$

$$\sin\alpha = \alpha - \frac{\alpha^3}{3!} + \cdots$$

$$\sin\theta_1\left(1-\frac{\delta^2}{2}\right) - \frac{\sin^2\theta_1}{\left(1-\frac{\delta^2}{2}\right)}\delta + \gamma\frac{\sin\theta_1}{\cos\theta_1} \doteq \sin\theta_1\left(1-\frac{\delta^2}{2}\right) + \cos\theta_1\delta$$

$$-\frac{\sin^2\theta_1}{1-\frac{\delta^2}{2}}\delta + \gamma\frac{\sin\theta_1}{\cos\theta_1} \doteq \cos\theta_1\delta$$

$$\frac{-\sin^2\theta_1\cos\theta_1\delta + \gamma\sin\theta_1\left(1-\frac{\delta^2}{2}\right)}{\left(1-\frac{\delta^2}{2}\right)\cos\theta_1} \doteq \cos\theta_1\delta$$

$$-\sin^2\theta_1\cos\theta_1\delta + \gamma\sin\theta_1 - \gamma\sin\theta_1\frac{\delta^2}{2} \doteq \cos^2\theta_1\delta\left(1-\frac{\delta^2}{2}\right)$$

$$\doteq \cos^2\theta_1\delta - \cos^2\theta_1\frac{\delta^2}{2}$$

$$-\gamma\sin\theta_1\frac{\delta^2}{2} + \cos^2\theta_1\frac{\delta^2}{2} - \sin^2\theta_1\cos\theta_1\delta - \cos^2\theta_1\delta + \gamma\sin\theta_1 \doteq 0$$

$$\frac{\cos^2\theta_1 - \gamma\sin\theta_1}{2}\delta^2 - (\sin^2\theta_1\cos\theta_1 + \cos^2\theta_1)\delta + \gamma\sin\theta_1 \doteq 0$$

$$\boxed{\delta \doteq \frac{\sin^2\theta_1\cos\theta_1 + \cos^2\theta_1 \pm \left[(\sin^2\theta_1\cos\theta_1 + \cos^2\theta_1)^2 - 4\left(\frac{\cos^2\theta_1 - \gamma\sin\theta_1}{2}\right)\gamma\sin\theta_1\right]^{1/2}}{\cos^2\theta_1 - \gamma\sin\theta_1}}$$

For a finite mass of m_2, $\gamma \to 0$. Then

$$\delta \doteq \frac{\sin^2\theta_1 \cos\theta_1 + \cos^2\theta_1 \pm \left[(\sin^2\theta_1 \cos\theta_1 + \cos^2\theta_1)^2 - 0\right]^{1/2}}{\cos^2\theta_1}$$

$$\delta \doteq \frac{\sin^2\theta_1 \cos\theta_1 + \cos^2\theta_1 \pm \sin^2\theta_1 \cos\theta_1 + \cos^2\theta_1}{\cos^2\theta_1}$$

Taking the difference gives

$$\delta \doteq 0$$

the sum has no physical significance since δ must be small.

4-35 A pair of masses m_1, m_2 connected by a spring of force constant K, slide without friction along the x-axis. Show that the center of mass moves with uniform velocity and the masses oscillate with frequency $\left[K\left(\frac{m_1+m_2}{m_1 m_2}\right)\right]^{1/2}$.

This is a two body problem.

$$M\ddot{\vec{R}} = \vec{F}_{ext.} \quad \{\text{center of mass}\}$$

$$\mu\ddot{\vec{r}} = F_{int} \quad \{\text{relative motion}\}$$

Since the external net forces are zero {horizontal frictionless motion}

$$M\ddot{\vec{R}} = \vec{O}$$

$$\ddot{R} = 0$$

$$\dot{R} = \text{constant} \quad \{\text{c.m. moving at uniform velocity}\}$$

The relative motion equation is a one body motion of reduced mass μ.

$$\mu\ddot{x} = -Kx \qquad \mu = \frac{m_1 m_2}{m_1+m_2}$$

$$\mu\ddot{x} + Kx = 0$$

$$\ddot{x} + \frac{K}{\mu}x = 0 \qquad x = Ae^{\lambda t}$$

$$\lambda^2 + \frac{K}{\mu} = 0$$

$$\lambda^2 = -\frac{K}{\mu}$$

$$\lambda = \pm i\left(\frac{K}{\mu}\right)^{1/2}$$

$$x = Ae^{\pm i\left(\frac{K}{\mu}\right)^{1/2} t}$$

$$= A\cos\left(\left(\frac{K}{\mu}\right)^{1/2} t + \theta_1\right)$$

$$\omega = \left(\frac{K}{\mu}\right)^{1/2}$$

$$\omega = \left[K\,\frac{m_1+m_2}{m_1 m_2}\right]^{1/2}$$

4-37 For the normal mode of vibration given by Eq. 4.162 and 4.163, find the force exerted on m_1 through the coupling spring, and show that the motion of x_1 satisfies the equation for a simple harmonic oscillator subject to a driving force.

$$x_1 = A_1 \cos(\omega_1 t + \theta_1) \qquad \{\text{Eq. } 4.162\}$$

$$x_2 = \frac{\Delta\omega^2}{2K^2}\sqrt{\frac{m_1}{m_2}}\, A_1 \cos(\omega_1 t + \theta_1) \qquad \{\text{Eq. } 4.163\}$$

The oscillation of m_1 and m_2 are out of phase. The force on m_1 through the coupling spring is

$$\therefore F = -k_3(x_2 + x_1)$$

$$\boxed{F = -k_3\left(A_1 \cos(\omega_1 t + \theta_1)\left[1 + \frac{\Delta\omega^2}{2K^2}\sqrt{\frac{m_1}{m_2}}\right]\right)}$$

The equation of motion of m_1 is

$$m_1 \ddot{x}_1 = F - kx_1$$

$$m_1 \ddot{x}_1 + kx_1 = F$$

This is the equation for a simple harmonic oscillator subject to a driving force.

1-39. Find the two normal modes of vibration for a pair of identical damped coupled harmonic oscillators {Eqs. 4.180 and 4.181}. That is, $m_1 = m_2, b_1 = b_2, K_1 = K_2$.

The equations of motion are:

$$m\ddot{x}_1 = -Kx_1 - K_3(x_1+x_2) - b\dot{x}_1 \qquad m_1 = m_2 = m$$

$$m\ddot{x}_2 = -Kx_2 - K_3(x_1+x_2) - b\dot{x}_2$$

$$m\ddot{x}_1 + b\dot{x}_1 + K'x_1 + K_3x_2 = 0 \qquad K' = K + K_3$$

$$m\ddot{x}_2 + b\dot{x}_2 + K'x_2 + K_3x_1 = 0$$

Let

$$x_1 = C_1 e^{pt}$$

$$x_2 = C_2 e^{pt}$$

$$\ddot{x}_1 = p^2 C_1 e^{pt} \qquad \dot{x}_1 = pC_1 e^{pt}$$

$$\ddot{x}_2 = p^2 C_2 e^{pt} \qquad \dot{x}_2 = pC_2 e^{pt}$$

$$mC_1p^2 + bC_1p + K'C_1 + K_3C_2 = 0$$

$$mC_2p^2 + bC_2p + K'C_2 + K_3C_1 = 0$$

$$\frac{C_2}{C_1} = -\frac{mp^2+bp+K'}{K_3} = -\frac{K_3}{mp^2+bp+K'}$$

$$m^2p^4 + mbp^3 + mK'p^2 + mbp^3 + b^2p^2 + bK'p + mK'p^2 + bK'p + K'^2 - K_3^2 = 0$$

$$m^2p^4 + 2mbp^3 + (2mK' + b^2)p^2 + 2bK'p + K'^2 - K_3^2 = 0$$

$$p^4 + \frac{2b}{m}p^3 + \left(\frac{2K'}{m} + \frac{b^2}{m^2}\right)p^2 + \frac{2bK'}{m^2}p + \frac{K'^2 - K_3^2}{m^2} = 0$$

The general form of the roots are:

$$p = -\gamma_1 \pm i\omega_1$$

$$p = -\gamma_2 \pm i\omega_2$$

For an uncoupled damped oscillator the solution is

$$p = -\gamma \pm i\omega_1 \qquad \gamma = \frac{b}{2m}, \ \omega_1 = (\omega_0^2 - \gamma^2)^{1/2} \quad \{K_3 = 0\}$$

This solution is obtained from

$$mp^2 + bp + k = 0$$

$$p^2 + \frac{b}{m}p + \frac{k}{m} = 0$$

$$p = -\frac{b}{2m} \pm \left[\left(\frac{b}{2m}\right)^2 - \frac{k}{m}\right]^{1/2}$$

Note also that the constant term, $\frac{k}{m}$, is a possible factor of

$$p^2 + \frac{b}{m}p + \frac{k}{m} = 0$$

Extending this to the equation

$$p^4 + \frac{2b}{m}p^3 + \left(\frac{2K}{m} + \frac{b^2}{m^2}\right)p^2 + \frac{2bk'}{m^2}p + \frac{k'^2 - k_3^2}{m^2} = 0$$

gives possible factors of

$$\frac{k'^2 - k_3^2}{m^2} = \frac{k'+k}{m} \quad \frac{k'-k}{m}$$

Therefore, by analogy

$$p = -\gamma \pm i\omega_1 \qquad \omega_1^2 = -\gamma^2 + \frac{k'+k}{m}$$

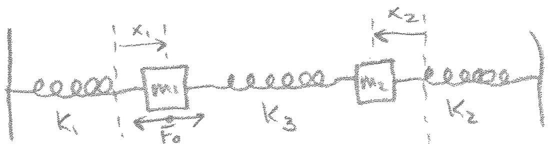
$$p = -\gamma \pm i\omega_2 \qquad \omega_2^2 = -\gamma^2 + \frac{k'-k}{m}$$

The normal modes are:

$$x_1 = x_2 = Ae^{-\gamma t}\cos(\omega_1 t + \theta_1)$$

$$x_1 = -x_2 = Ae^{-\gamma t}\cos(\omega_2 t + \theta_2)$$

4-41 An applied force $F = F_0 \cos\omega t$ is applied to m_1. Set up the equations of motion and find the steady state solution. Sketch the amplitude and phase of each oscillator as functions of ω.



$$m_1\ddot{x}_1 = -K_1x_1 - K_3(x_1+x_2) + F_0\cos\omega t$$

$$m_2\ddot{x}_2 = -K_2x_2 - K_3(x_1+x_2)$$

$$m_1\ddot{x}_1 + K_1'x_1 + K_3x_2 = F_0\cos\omega t \quad , \; K_1' = K_1 + K_3$$

$$m_2\ddot{x}_2 + K_2'x_2 + K_3x_1 = 0 \quad , \; K_2' = K_2 + K_3$$

$$x_1 = C_1e^{pt}$$
$$x_2 = C_2e^{pt}$$

$$(m_1p^2 + K_1')C_1 + K_3C_2 = 0 \qquad \{\text{homogeneous equations}\}$$

$$(m_2p^2 + K_2')C_2 + K_3C_1 = 0$$

$$\frac{C_2}{C_1} = -\frac{m_1p^2+K_1'}{K_3} = -\frac{K_3}{m_2p^2+K_2'}$$

$$\frac{m_1p^2+K_1'}{K_3} = \frac{K_3}{m_2p^2+K_2'}$$

$$m_1m_2p^4 + (m_2K_1' + m_1K_2')p^2 + (K_1'K_2' - K_3^2) = 0$$

$$p^2 = \frac{-(m_2K_1'+m_1K_2') \pm \left[(m_2K_1'+m_1K_2')^2 - 4m_1m_2(K_1'K_2'-K_3^2)\right]^{1/2}}{2m_1m_2}$$

$$p^2 = -\frac{1}{2}\left(\frac{K_1'}{m_1}+\frac{K_2'}{m_2}\right) \pm \left[\frac{1}{4}\left(\frac{K_1'}{m_1}+\frac{K_2'}{m_2}\right)^2 - \frac{K_1'K_2'}{m_1m_2} + \frac{K_3^2}{m_1m_2}\right]^{1/2}$$

Let $\omega_{10}^2 = \frac{K_1'}{m_1}$, $\omega_{20}^2 = \frac{K_2'}{m_2}$

$$p^2 = -\frac{1}{2}(\omega_{10}^2 + \omega_{20}^2) \pm \left[\frac{1}{4}(\omega_{10}^2 - \omega_{20}^2)^2 + \frac{K_3^2}{m_1m_2}\right]^{1/2}$$

Assume $\omega_{10}^2 \geq \omega_{20}^2$

then the quantity inside the square root brackets is always less than the square of the first term. Thus p^2 is always negative

$$p^2 = -\omega_1^2 = -(\omega_{10}^2 + \tfrac{1}{2}\Delta\omega^2)$$

$$p^2 = -\omega_2^2 = -(\omega_{20}^2 - \tfrac{1}{2}\Delta\omega^2)$$

$$\Delta\omega^2 = (\omega_{10}^2 - \omega_{20}^2)\left[\left(1 + \frac{4K^2}{(\omega_{10}^2-\omega_{20}^2)^2}\right)^{1/2} - 1\right] \quad \{\text{Eq. } 4.150\}$$

$$K^2 = \frac{K_3}{\sqrt{m_1 m_2}}$$

The four solutions are

$$p = \pm i\omega_1$$

$$p = \pm i\omega_2$$

when $p^2 = -\omega_1^2$

$$\frac{C_2}{C_1} = \frac{m_1(\omega_1^2 - \omega_{10}^2)}{K_3} = \frac{\Delta\omega^2}{2K^2}\sqrt{\frac{m_1}{m_2}}$$

when $p^2 = -\omega_2^2$

$$\frac{C_1}{C_2} = \frac{m_2}{K_3}(\omega_2^2 - \omega_{20}^2) = -\frac{\Delta\omega^2}{2K^2}\sqrt{\frac{m_2}{m_1}}$$

The general solution is the sum of the four solutions

$$x = x_1 + x_2$$

$$= C_1 e^{pt} + C_2 e^{pt}$$

$$x_1 = C_1 e^{i\omega_1 t} + C_1' e^{-i\omega t} - \frac{\Delta\omega^2}{2K^2}\sqrt{\frac{m_2}{m_1}} C_2 e^{i\omega_2 t} - \frac{\Delta\omega^2}{2K^2}\sqrt{\frac{m_2}{m_1}} e^{-i\omega_2 t}$$

$$x_2 = \frac{\Delta\omega^2}{2K^2}\sqrt{\frac{m_1}{m_2}} C_2 e^{i\omega_1 t} + \frac{\Delta\omega^2}{2K^2}\sqrt{\frac{m_1}{m_2}} e^{-i\omega_1 t} + C_2 e^{i\omega_2 t} + C_2' e^{-i\omega_2 t}$$

For the solutions to be real set

$$C_1 = \tfrac{1}{2}A_1 e^{i\theta_1} \qquad C_1' = \tfrac{1}{2}A_1 e^{-i\theta_1}$$

$$C_2 = \tfrac{1}{2}A_2 e^{i\theta_2} \qquad C_2' = \tfrac{1}{2}A_2 e^{-i\theta_2}$$

Then

$$x_1 = A_1\cos(\omega_1 t+\theta_1) - \frac{\Delta\omega^2}{2K^2}\sqrt{\frac{m_2}{m_1}}\,A_2\cos(\omega_2 t+\theta_2)$$

$$x_2 = A_2\cos(\omega_2 t+\theta_1) + \frac{\Delta\omega^2}{2K^2}\sqrt{\frac{m_1}{m_2}}\,A_1\cos(\omega_2 t+\theta_1)$$

These are the homogeneous solutions.

The equations of motion in matrix form is

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1' & K_3 \\ K_3 & K_2' \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} F_0\cos\omega t \\ 0 \end{bmatrix}$$

Let

$$x_1 = u_1\cos\omega t$$

$$x_2 = u_2\cos\omega t \quad \{m_2 \text{ will be forced by } F_0\cos\omega t\}$$

$$\ddot{x}_1 = -u_1\omega^2\cos\omega t$$

$$\ddot{x}_2 = -u_2\omega^2\cos\omega t$$

The equations of motion are

$$-m_1\omega^2 u_1 + K_1' u_1 + K_3 u_2 = F_0$$

$$-m_2\omega^2 u_2 + K_2' u_2 + K_3 u_1 = 0$$

In matrix form

$$\begin{bmatrix} K_1'-m_1\omega^2 & K_3 \\ K_3 & K_2'-m_2\omega^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$\text{Let } [S] = \begin{bmatrix} K_1'-m_1\omega^2 & K_3 \\ K_3 & K_2'-m_2\omega^2 \end{bmatrix}$$

Then

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = [S]^{-1}\begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

The inverse of $[S]$ is

$$[S]^{-1} = \frac{\operatorname{adj}[S]}{\det[S]} \quad \{\text{from matrix algebra}\}$$

$$\operatorname{adj}[S] = \begin{bmatrix} k_2' - m_2\omega^2 & -k_3 \\ -k_3 & k_1' - m_1\omega^2 \end{bmatrix}$$

$$\det[S] = (k_1' - m_1\omega^2)(k_2' - m_2\omega^2) - k_3^2$$

Now,

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{(k_1' - m_1\omega^2)(k_2' - m_2\omega^2) - k_3^2} \begin{bmatrix} k_2' - m_2\omega^2 & -k_3 \\ -k_3 & k_1' - m_1\omega^2 \end{bmatrix} \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

$$u_1 = \frac{(k_2' - m_2\omega^2)F_0}{(k_1' - m_1\omega^2)(k_2' - m_2\omega^2) - k_3^2} = \frac{(m_2\omega^2 - k_2')F_0}{k_3^2 - (m_1\omega^2 - k_1')(m_2\omega^2 - k_2')}$$

$$u_2 = \frac{-k_3F_0}{(k_1' - m_1\omega^2)(k_2' - m_2\omega^2) - k_3^2} = \frac{k_3F_0}{k_3^2 - (m_1\omega^2 - k_1')(m_2\omega^2 - k_2')}$$

$$\left.\begin{aligned} x_1 &= \frac{(m_2\omega^2 - k_2')F_0\cos\omega t}{k_3^2 - (m_1\omega^2 - k_1')(m_2\omega^2 - k_2')} \\ x_2 &= \frac{k_3F_0\cos\omega t}{k_3^2 - (m_1\omega^2 - k_1')(m_2\omega^2 - k_2')} \end{aligned}\right\} \text{steady state solutions}$$

The general solutions are

$$x_1 = A_1\cos(\omega_1 t + \theta_1) - \frac{\Delta\omega}{2K^2}\sqrt{\frac{m_1}{m_2}}A_2\cos(\omega_2 t + \theta_2) + \frac{(m_2\omega^2 - k_2')F_0\cos\omega t}{k_3^2 - (m_1\omega^2 - k_1')(m_2\omega^2 - k_2')}$$

$$x_2 = A_2\cos(\omega_2 t + \theta_2) + \frac{\Delta\omega}{2K^2}\sqrt{\frac{m_2}{m_1}}A_1\cos(\omega_1 t + \theta_1) + \frac{k_3F_0\cos\omega t}{k_3^2 - (m_1\omega^2 - k_1')(m_2\omega^2 - k_2')}$$

Sketch of amplitude vs. ω

For m_1:

$$X_{1s} = \frac{(m_2\omega^2 - k_2')F_0\cos\omega t}{K_3^2 - (m_1\omega^2 - k_1')(m_2\omega^2 - k_2')} \qquad k_1' = m_1\omega_{10}^2,\ k_2' = m_2\omega_{20}^2$$

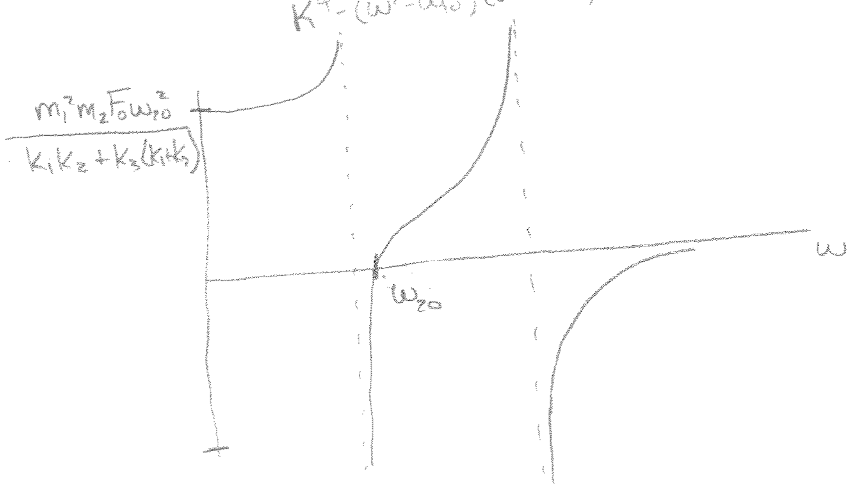
$$= \frac{(m_2\omega^2 - m_2\omega_{20}^2)F_0\cos\omega t}{K_3^2 - (m_1\omega^2 - m_1\omega_{10}^2)(m_2\omega^2 - m_2\omega_{20}^2)}$$

$$= \frac{m_2(\omega^2 - \omega_{20}^2)F_0\cos\omega t}{K_3^2 - m_1(\omega^2 - \omega_{10}^2)m_2(\omega^2 - \omega_{20}^2)}$$

$$= \frac{m_2(\omega^2 - \omega_{20}^2)F_0\cos\omega t}{\frac{K_3^2}{m_1m_2} - (\omega^2 - \omega_{10}^2)(\omega^2 - \omega_{20}^2)} \quad \frac{1}{m_1m_2}$$

$$X_{1s} = \frac{m_1(\omega^2 - \omega_{20}^2)F_0\cos\omega t}{K^4 - (\omega^2 - \omega_{10}^2)(\omega^2 - \omega_{20}^2)} \qquad \left\{ K^2 = \frac{K_3}{\sqrt{m_1m_2}} \right\}$$

$$A_1 = \frac{m_1(\omega^2 - \omega_{20}^2)F_0}{K^4 - (\omega^2 - \omega_{10}^2)(\omega^2 - \omega_{20}^2)}$$



steady state amplitude vs. ω
Mass 1

when $\omega = 0$

$$A_1 = \frac{-m_1F_0\omega_{20}^2}{K^4 - (\omega_{10}^2)(\omega_{20}^2)}$$

$$= \frac{-m_1F_0\omega_{20}^2}{\frac{K_3^2}{m_1m_2} - \left(\frac{k_1'}{m_1}\right)\left(\frac{k_2'}{m_2}\right)}$$

$$= \frac{-m_1F_0\omega_{20}^2}{\frac{1}{m_1m_2}(K_3^2 - k_1'k_2')}$$

$$= \frac{-m_1F_0\omega_{20}^2}{\frac{1}{m_1m_2}(K_3^2 - (k_1+k_3)(k_2+k_3))}$$

$$= \frac{-m_1F_0\omega_{20}^2}{\frac{1}{m_1m_2}(K_3^2 - (k_1k_2 + k_3(k_1+k_2) + k_3^2))}$$

$$= \frac{-m_1F_0\omega_{20}^2}{\frac{1}{m_1m_2}(-k_1k_2 - k_3(k_1+k_2))}$$

$$= \frac{m_1F_0\omega_{20}}{\frac{1}{m_1m_2}(k_1k_2 + k_3(k_1+k_2))}$$

For m_2:

$$X_{2s} = \frac{k_3 F_0 \cos\omega t}{k_3^2 - (m_1\omega^2 - k_1')(m_2\omega^2 - k_2')}$$

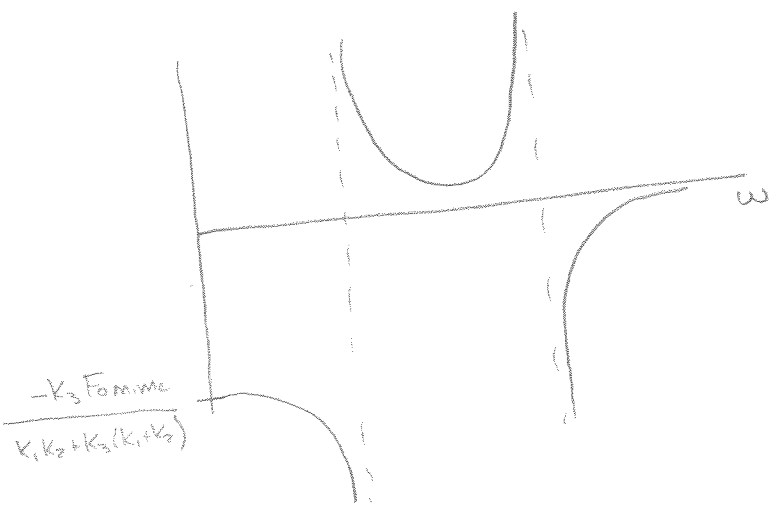
$$X_{2s} = \frac{k_3 F_0 \cos\omega t}{K^4 - (\omega^2 - \omega_{10}^2)(\omega^2 - \omega_{20}^2)}$$

$$A_2 = \frac{k_3 F_0}{K^4 - (\omega^2 - \omega_{10}^2)(\omega^2 - \omega_{20}^2)}$$

when $\omega = 0$

$$A_2 = \frac{k_3 F_0}{K^4 - (\omega^2 - \omega_{10}^2)(\omega^2 - \omega_{20}^2)}$$

$$= \frac{-k_3 F_0 m_1 m_2}{k_1 k_2 + k_3(k_1 + k_2)}$$



Steady state amplitude vs. ω
Mass 2

5-1

a) For a system of particles making up a rigid body

$$T = \sum_{K=1}^{N} \frac{1}{2} m_K v_K^2 \quad \{Eq. 4.37\}$$

For a rigid body rotating about an axis, the moment of inertia of the rigid body made from a system of particles is

$$I_z = \sum_{K=1}^{N} m_K r_K^2$$

The tangential velocity is

$$v = r\dot{\theta}$$

$$v^2 = r^2 \dot{\theta}^2$$

For a system of particles

$$v_K^2 = r_K^2 \dot{\theta}^2 \quad \{\dot{\theta} \text{ is the same for each particle of the rigid body}\}$$

Substituting v_K^2 into the kinetic energy equation gives

$$T = \sum_{K=1}^{N} \frac{1}{2} m_K r_K^2 \dot{\theta}^2$$

$$= \frac{1}{2}\left(\sum_{K=1}^{N} m_K r_K^2\right)\dot{\theta}^2$$

$$T = \frac{1}{2} I_z \dot{\theta}^2 \quad \{Eq. 5.16\}$$

b) The definition of work is

$$V = -\int_{x_0}^{x} \vec{F} \cdot d\vec{x}$$

where $\vec{F}$ is the sum of external forces.

Torque is defined as

$$\vec{N}_z = \vec{F}_\theta r$$

$$N_z = \vec{F} \cdot \vec{r}_\perp$$

$$N_z = F_\theta r$$

Now,

$$d\vec{r}_\perp = r\,d\theta\,\hat{\theta}$$

$$d\vec{x} = d\vec{r}_\perp$$
$$= r\,d\theta\,\hat{\theta}$$

Substituting into the definition of work gives

$$V = -\int_{\theta_s}^{\theta} \vec{F} \cdot r\,d\theta\,\hat{\theta}$$

$$= -\int_{\theta_s}^{\theta} F_\theta\, r\, d\theta$$

$$V = -\int_{\theta_s}^{\theta} N_z\, d\theta \qquad \{Eq.\ 5.14\}$$

5-3

Starting with

$$\frac{dL}{dt} = I_z \ddot{\theta} = N_z$$

where N_z is a function of θ.

The energy theorm is

$$T_2 - T_1 = \int_{x_1}^{x_2} F\,dx$$

$$I_z \frac{d\dot{\theta}}{dt} = N_z$$

$$\frac{1}{2} I_z \dot{\theta}^2 - \frac{1}{2} I_z \dot{\theta}_0^2 = \int_{\theta_0}^{\theta} N_z\,d\theta$$

The potential energy is defined as

$$V = -\int_{x_s}^{x} F\,dx$$

$$= -\int_{\theta_0}^{\theta} N_z\,d\theta$$

$$-\int_{\theta_0}^{\theta} N_z\,d\theta = -V(\theta) + V(\theta_0)$$

Now,

$$\frac{1}{2} I_z \dot{\theta}^2 - \frac{1}{2} I_z \dot{\theta}_0^2 = -V(\theta) + V(\theta_0)$$

$$\frac{1}{2} I_z \dot{\theta}^2 + V(\theta) = \frac{1}{2} I_z \dot{\theta}_0^2 + V(\theta_0)$$

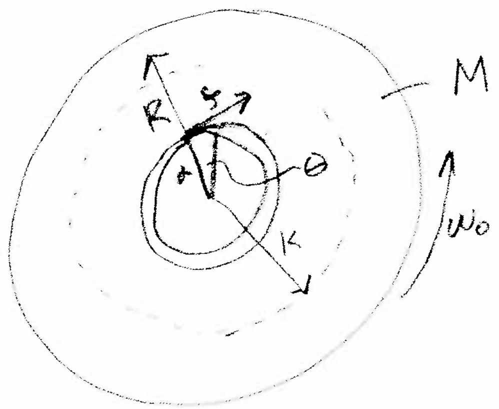
The quantity on the right only depends on the initial conditions and is therefore constant. So

$$T + V = \text{constant}.$$

5-

5. A wheel of mass M, radius of gyration k, spins smoothly on a *fixed* horizontal axle of radius a which passes through a hole of slightly larger radius at the hub of the wheel. The coefficient of friction between the bearing surfaces is μ. If the wheel is initially spinning with angular velocity ω_0, find the time and the number of turns that it takes to stop.

Number of Turns (time on back side)



$f = \mu M g \cos\theta$

Work-Energy Theorem:

$\Delta K = W$

$\Delta K = \frac{1}{2} I \omega_0^2 - \frac{1}{2} I (0)$

$\Delta K = \frac{1}{2} I \omega_0^2$

$I = M K^2$

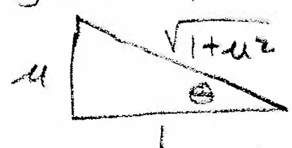
$\Delta K = \frac{M K^2 \omega_0^2}{2}$

$W = T\theta$ (T = torque)

$= f a \theta$

$= \mu a \cos\theta \, M g \, \theta$

$\mu = \tan\theta$ {angle of repose}



$\cos\theta = \frac{1}{\sqrt{1+\mu^2}}$

$$\frac{M K^2 \omega_0^2}{2} = \mu M g a \theta \cos\theta$$

$$\frac{K^2 \omega_0^2}{2} = \mu g a \theta \frac{1}{\sqrt{1+\mu^2}}$$

$$\theta = \frac{K^2 \omega_0^2 \sqrt{1+\mu^2}}{2 \mu g a}$$

$$\boxed{\# \text{of turns} = \frac{\theta}{2\pi} = \frac{K^2 \omega_0^2 \sqrt{1+\mu^2}}{4\pi \mu g a}}$$

$$\frac{dL}{dt} = \Gamma$$

$$dL = \Gamma\, dt$$

$$\int_0^{\omega_0} dL = \Gamma \int_0^{t_s} dt$$

$$L\Big|_0^{\omega_0} = \Gamma\, t\Big|_0^{t_s}$$

$$Mk^2\omega_0 = \frac{\mu M g a}{\sqrt{1+\mu^2}}\, t_s$$

$$\boxed{\frac{k^2\omega_0\sqrt{1+\mu^2}}{\mu g a} = t_s}$$

$$\Gamma = f a$$

$$= \mu M g a \cos\theta$$

$$= \frac{\mu M g a}{\sqrt{1+\mu^2}}$$

$$L = I\dot{\theta}$$

$$= Mk^2\dot{\theta}$$

5-7

$$I\ddot{\theta} = N + N_f \qquad N = N_0(1+\alpha\cos\omega_0 t)$$
$$N_f = -b\dot{\theta}$$

$$I\ddot{\theta} = N_0 + N_0\alpha\cos\omega_0 t - b\dot{\theta} \rightarrow I\ddot{\theta} + b\dot{\theta} = N_0 + N_0\alpha\cos\omega_0 t$$

Eqn. 1 $I\ddot{\theta} = N_0 - b\dot{\theta}$

$$I\frac{d\dot{\theta}}{dt} = N_0 - b\dot{\theta}$$

$$\frac{d\dot{\theta}}{N_0 - b\dot{\theta}} = \frac{dt}{I}$$

$$\int_0^{\dot{\theta}} \frac{d\dot{\theta}}{N_0 - b\dot{\theta}} = \int_0^t \frac{dt}{I}$$

$$u = N_0 - b\dot{\theta}$$
$$du = +b\,d\dot{\theta}$$
$$-\frac{du}{b} = d\dot{\theta}$$

$$-\frac{1}{b}\int \frac{du}{u} = -\frac{1}{b}\ln|u| + C$$

$$\int_0^{\dot{\theta}} \frac{d\dot{\theta}}{N_0 - b\dot{\theta}} = -\frac{1}{b}\ln|N_0 - b\dot{\theta}|\Big|_0^{\dot{\theta}}$$
$$= -\frac{1}{b}\ln\frac{N_0 - b\dot{\theta}}{N_0}$$
$$= -\frac{1}{b}\ln\left(1 - \frac{b\dot{\theta}}{N_0}\right)$$

$$-\frac{1}{b}\ln\left(1 - \frac{b\dot{\theta}}{N_0}\right) = \frac{t}{I}$$

$$\ln\left(1 - \frac{b\dot{\theta}}{N_0}\right) = -\frac{b}{I}t$$

$$1 - \frac{b\dot{\theta}}{N_0} = e^{-\frac{b}{I}t}$$

$$\dot{\theta}\frac{b}{N_0} = 1 - e^{-\frac{b}{I}t}$$

$$\dot{\theta} = \frac{N_0}{b}\left(1 - e^{-\frac{b}{I}t}\right)$$

For steady state, $t \to \infty$

$$\dot{\theta}_{s_1} = \frac{N_0}{b}$$

Eqn. 2 $I\ddot{\theta} = N_0\alpha\cos\omega_0 t - b\dot{\theta}$

$$\ddot{\theta} + \frac{b}{I}\dot{\theta} = \frac{N_0\alpha}{I}\cos\omega_0 t$$

$$\theta_p = A\cos\omega_0 t + B\sin\omega_0 t$$
$$\dot{\theta}_p = -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t$$
$$\ddot{\theta}_p = -A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t$$

$$-A\omega_0^2\cos\omega_0 t - B\omega_0^2\sin\omega_0 t + \frac{b}{I}\left(-A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t\right) =$$
$$\frac{N_0\alpha}{I}\cos\omega_0 t$$

Equating like terms:

1) $-A\omega_0^2 + B\frac{b}{I}\omega_0 = \frac{N_0\alpha}{I}$

2) $-B\omega_0^2 - A\frac{b}{I}\omega_0 = 0 \rightarrow B = -\frac{Ab}{I\omega_0}$

$$-A\omega_0^2 - A\frac{b^2}{I^2} = \frac{N_0\alpha}{I}$$
$$-A\left(\omega_0^2 - \frac{b^2}{I^2}\right) = \frac{N_0\alpha}{I}$$
$$-A\left(\frac{I^2\omega_0^2 - b^2}{I^2}\right) = \frac{N_0\alpha}{I}$$

$$A = \frac{-N_0\alpha I}{I^2\omega_0^2 - b^2}$$

$$B = \frac{N_0\alpha b}{\omega_0(I^2\omega_0^2 - b^2)}$$

$$\theta_p = \frac{-N_0\alpha I}{I^2\omega_0^2 - b^2}\cos\omega_0 t + \frac{N_0\alpha b}{I\omega_0(I^2\omega_0^2 - b^2)}\sin\omega_0 t$$

The homogeneous equation $\rightarrow 0$ as $t \to \infty$, of the form e^{-kt}.

$$\boxed{\theta_{s_2} = \frac{N_0\alpha b}{\omega_0(I^2\omega_0^2 - b^2)}\sin\omega_0 t - \frac{N_0\alpha I}{I^2\omega_0^2 - b^2}\cos\omega_0 t}$$

$$d\theta_{s_1} = \frac{N_0}{b}\,dt$$

$$\int_{\theta_0}^{\theta_s} d\theta_{s_1} = \int_0^t \frac{N_0}{b}\,dt$$

$$\theta_{s_1} - \theta_0 = \frac{N_0}{b}t$$

$$\boxed{\theta_{s_1} = \theta_0 + \frac{N_0 t}{b}}$$

$$\theta_s = \theta_{s_1} + \theta_{s_2}$$

$$= \theta_0 + \frac{N_0}{b}t + \frac{N_0 \alpha b}{\omega_0(I^2\omega_0^2 - b^2)}\sin\omega_0 t - \frac{N_0 \alpha I}{I^2\omega_0^2 - b^2}\cos\omega_0 t$$

$$\boxed{\theta_s = \theta_0 + \frac{N_0}{b}t + \frac{\alpha N_0}{I^2\omega_0^3 - b^2\omega_0}\left[b\sin\omega_0 t - I\omega_0\cos\omega_0 t\right]}$$

5-9

i) Solve for ϕ

$$\phi + \frac{1}{8}a^2(2\phi - \sin 2\phi) + \cdots = \left(\frac{g}{\ell}\right)^{1/2} t \quad \{\text{Eq. } 5.33\}$$

$$\phi \doteq \left(\frac{g}{\ell}\right)^{1/2} t - \frac{1}{8}a^2(2\phi - \sin 2\phi)$$

$$\doteq \omega_0 t - \frac{a^2}{4}\phi + \frac{a^2}{8}\sin 2\phi \quad \left\{\omega_0 = \left(\frac{g}{\ell}\right)^{1/2}\right\}$$

First approximation

$$\phi^{(1)} = \omega_0 t$$

second approximation

$$\phi^{(2)} = \omega_0 t - \frac{a^2}{4}\phi^{(1)} + \frac{a^2}{8}\sin 2\phi^{(1)}$$

$$\phi^{(2)} = \omega_0 t - \frac{a^2}{4}\omega_0 t + \frac{a^2}{8}\sin 2\omega_0 t \qquad a = \sin\frac{K}{2} \doteq \frac{K}{2} - \frac{K^3}{8 \cdot 3!} = \frac{K}{2} - \frac{K^3}{48}$$

$$= \omega_0 t - \frac{1}{4}\left(\frac{K}{2} - \frac{K^3}{48}\right)^2 \omega_0 t + \frac{1}{8}\left(\frac{K}{2} - \frac{K^3}{48}\right)^2 \sin 2\omega_0 t$$

$$= \omega_0 t - \frac{1}{4}\left(\frac{K^2}{4} + \text{terms} > K^3\right)\omega_0 t + \frac{1}{8}\left(\frac{K^2}{4} + \text{terms} > K^3\right)\sin 2\omega_0 t$$

$$\doteq \omega_0 t - \frac{K^2}{16}\omega_0 t + \cdots$$

$$\boxed{\phi^{(2)} = \omega_0 t\left(1 - \frac{K^2}{16} + \cdots\right) \longrightarrow \omega' = \omega_0\left(1 - \frac{K^2}{16} + \cdots\right) \quad \left\{\omega_0 = \left(\frac{g}{\ell}\right)^{1/2}\right\}}$$

ii) Solve for θ

$$\frac{1}{a}\sin\frac{\theta}{2} = \sin\phi \quad \{\text{Eq. } 5.29\}$$

$$\sin\frac{\theta}{2} = a\sin\phi$$

$$\frac{\theta}{2} - \frac{\theta^3}{48} \doteq a\sin\phi$$

$$\theta = 2a\sin\phi + \frac{\theta^3}{24}$$

$$\theta^{(1)} = 2a\sin\phi$$

$$\theta^{(2)} = 2a\sin\phi + \frac{(2a\sin\phi)^3}{24}$$

$$= 2a\sin\phi + \frac{a^3}{3}\sin^3\phi \qquad \sin^3\phi = \frac{3\sin\phi - \sin 3\phi}{4}$$

$$= 2a\sin\phi + \frac{a^3}{3}\left[\frac{3\sin\phi - \sin 3\phi}{4}\right]$$

$$= 2\left(\frac{K}{2} - \frac{K^3}{48}\right)\sin\phi + \frac{\left(\frac{K}{2} - \frac{K^3}{48}\right)^3}{3}\left[\frac{3\sin\phi - \sin 3\phi}{4}\right]$$

$$= \left(K - \frac{K^3}{24}\right)\sin\phi + \frac{K^3}{24}\left[\frac{3\sin\phi - \sin 3\phi}{4}\right] \quad \left\{\left(\frac{K}{2} - \frac{K^3}{48}\right)^3 = \frac{K^3}{8} + \text{terms} > K^3\right\}$$

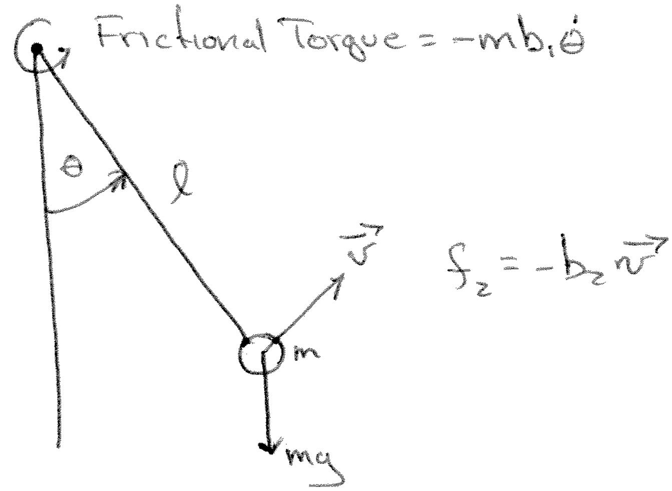
$$= \left(K - \frac{K^3}{24}\right)\sin\phi + \frac{K^3}{32}\sin\phi - \frac{K^3\sin 3\phi}{96}$$

$$= \left(K - \frac{K^3}{96}\right)\sin\phi - \frac{K^3}{96}\sin 3\phi$$

$$\boxed{\theta^{(2)} = \left(K - \frac{K^3}{96}\right)\sin\omega' t - \frac{K^3}{96}\sin 3\omega' t}$$

Note: equation 5.35 in text is different.

5-10



$I_z = ml^2$

$N_{z_g} = -mgl\sin\theta$ {torque due to gravity}

$N_{z_{f_1}} = -mb_1\dot{\theta}$

$N_{z_{f_2}} = -b_2\vec{v}l - b_2 l\dot{\theta} l = -b_2 l^2\dot{\theta}$ $\{v = r\dot{\theta} = l\dot{\theta}\}$

Equation of motion:

$$\frac{d\vec{L}}{dt} = I_z\ddot{\theta} = \sum_i N_z$$

$$= N_{z_g} + N_{z_{f_1}} + N_{z_{f_2}}$$

$$ml^2\ddot{\theta} = -mgl\sin\theta - mb_1\dot{\theta} - b_2 l^2\dot{\theta}$$

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{b_1}{l^2}\dot{\theta} - \frac{b_2}{m}\dot{\theta}$$

$$\doteq -\frac{g}{l}\theta - \left(\frac{b_1}{l^2} + \frac{b_2}{m}\right)\dot{\theta} \quad \{\sin\theta \doteq \theta \text{ for small } \theta\}$$

$$\ddot{\theta} + \left(\frac{b_1}{l^2} + \frac{b_2}{m}\right)\dot{\theta} + \frac{g}{l}\theta = 0 \qquad \theta = Ae^{\lambda t}$$

$$\lambda^2 + \lambda\left(\frac{b_1}{l^2} + \frac{b_2}{m}\right) + \frac{g}{l} = 0$$

$$\lambda = \frac{-\left(\frac{b_1}{l^2} + \frac{b_2}{m}\right) \pm \left[\left(\frac{b_1}{l^2} + \frac{b_2}{m}\right)^2 - \frac{4g}{l}\right]^{1/2}}{2}$$

$$\lambda = -\left(\frac{b_1}{2\ell^2} + \frac{b_2}{2m}\right) \pm \left[\frac{1}{4}\left(\frac{b_1}{\ell^2} + \frac{b_2}{m}\right)^2 - \frac{g}{\ell}\right]^{1/2}$$

Assume $\frac{g}{\ell} > \frac{1}{4}\left(\frac{b_1}{\ell^2} + \frac{b_2}{m}\right)^2$

$$\lambda = \underbrace{-\left(\frac{b_1}{2\ell^2} + \frac{b_2}{2m}\right)}_{\text{damping term}} \pm \underbrace{i\left[\frac{g}{\ell} - \frac{1}{4}\left(\frac{b_1}{\ell^2} + \frac{b_2}{m}\right)^2\right]^{1/2}}_{\text{oscillation term}}$$

$$\theta = A e^{-\left(\frac{b_1}{2\ell^2} + \frac{b_2}{2m}\right)t} \cos(\omega t + \theta_0)$$

$$\omega = \left[\frac{g}{\ell} - \frac{1}{4}\left(\frac{b_1}{\ell^2} + \frac{b_2}{m}\right)^2\right]^{1/2}$$

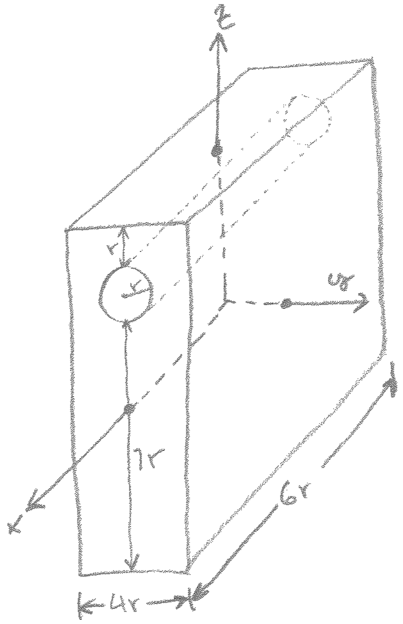
The amplitude A will be $A \cdot \frac{1}{e} = Ae^{-1}$ when

$$\left(\frac{b_1}{2\ell^2} + \frac{b_2}{2m}\right)t = 1$$

$$\left(\frac{2mb_1 + 2\ell^2 b_2}{4m\ell^2}\right)t = 1$$

$$\boxed{t = \frac{4m\ell^2}{2mb_1 + 2\ell^2 b_2}}$$

5-17



r = 1 cm

Find I_x, I_y, and I_z about the center of mass.

The x-y-z axes ~~origin is located~~ at the prism geometric center.
The parallel axis theorem will be utilized to determine I_x, I_y, and I_z.

First, the moment integrals will be found.

$dm = \rho\, dx\,dy\,dz$

$$X = \rho \int_{-5r}^{5r}\int_{-2r}^{2r}\int_{-3r}^{3r} x^2\,dx\,dy\,dz$$

$$= \rho(10r)(4r)\frac{x^3}{3}\Big|_{-3r}^{3r}$$

$$= \rho(10r)(4r)\left[\frac{27r^3}{3}+\frac{27r^3}{3}\right]$$

$$= \rho(10r)(4r)(18r^3)$$

$$X = 720\rho r^5$$

$$Y = \rho \int_{-5r}^{5r}\int_{-3r}^{3r}\int_{-2r}^{2r} y^2\,dy\,dx\,dz$$

$$= \rho(10r)(6r)\frac{y^3}{3}\Big|_{-2r}^{2r}$$

$$= \rho(10r)(6r)\left[\frac{8r^3}{3}+\frac{8r^3}{3}\right]$$

$$= \rho(10r)(6r)\left(\frac{16r^3}{3}\right)$$

$$Y = 320\rho r^5$$

$$Z = \rho \int_{-2r}^{2r}\int_{-3r}^{3r}\int_{-5r}^{5r} z^2\,dz\,dx\,dy$$

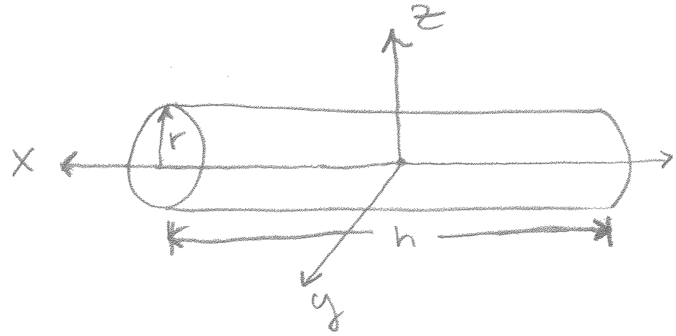
$$= \rho(4r)(6r)\frac{z^3}{3}\Big|_{-5r}^{5r}$$

$$= \rho(4r)(6r)\left[\frac{125r^3}{3}+\frac{125r^3}{3}\right]$$

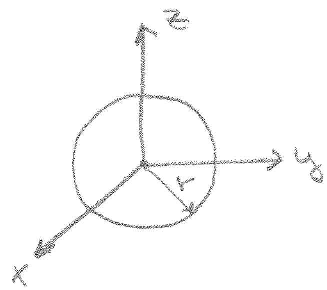
$$= \rho(4r)(6r)\left(\frac{250r^3}{3}\right)$$

$$Z = 2000\rho r^5$$

Now, find the moments for a cylinder



Begin with a lamina



$$I_{x\ell} = \int r'^2 dm \qquad dm = \rho r' dr' d\theta$$

$$= \rho \int_0^{2\pi} \int_0^r r'^3 dr' d\theta$$

$$= 2\pi \rho \frac{r^4}{4}$$

$$I_{x\ell} = \frac{\pi \rho r^4}{2}$$

$$I_{x_{sc}} = \int_{-\frac{h}{2}}^{h/2} I_{x\ell} \, dh'$$

$$= I_{x\ell} h$$

$$I_{x_{sc}} = \frac{\pi \rho r^4 h}{2}$$

Using the $\perp$ axis theorem $I_{z\ell}$ and $I_{y\ell}$ can be determined.

$$I_{x\ell} = I_{z\ell} + I_{y\ell}, \quad I_{z\ell} = I_{y\ell} \ \{\text{symmetry}\}$$

$$I_{x\ell} = 2 I_{z\ell}$$

$$I_{z\ell} = \frac{1}{2} I_{x\ell}$$

$$I_{z\ell} = \frac{\pi \rho r^4}{4} = I_{y\ell}$$

Now, using the // axis theorem, I_{zsc} can be found.

$$I = I_G + Mh'^2$$

$$I_{zsc} = \int_{-\frac{h}{2}}^{\frac{h}{2}} I_{z\ell}\, dh' + \int dm\, h'^2 \quad , \qquad dm = \rho r' dr' d\theta dh'$$

$$= \frac{\rho \pi r^4}{4} \int_{-\frac{h}{2}}^{\frac{h}{2}} dh' + \rho \int_{-\frac{h}{2}}^{\frac{h}{2}} \int_0^{2\pi} \int_0^{r} r' dr' d\theta\, h'^2 dh'$$

$$= \frac{\rho \pi r^4 h}{4} + \rho \pi r^2 \frac{h'^3}{3} \Big|_{-\frac{h}{2}}^{\frac{h}{2}}$$

$$= \frac{\rho \pi r^4 h}{4} + \frac{\rho \pi r^2}{3} \left[\frac{h^3}{8} + \frac{h^3}{8} \right]$$

$$I_{zsc} = \frac{\rho \pi r^4 h}{4} + \frac{\rho \pi r^2 h^3}{12}$$

$$I_{ysc} = I_{zsc}$$

$$I_{ysc} = \frac{\rho \pi r^4 h}{4} + \frac{\rho \pi r^2 h^3}{12}$$

The center of mass is to be found.

$$\bar{x} = \bar{y} = 0 \quad \{\text{symmetry}\}$$

$$\bar{z} = \frac{1}{M} \int r\, dm$$

$$= \frac{1}{M} \Big[(0r)(\rho)(10r)(4r)(6r) - (3r)\underbrace{(\rho)(\pi r^2)(6r)}_{\text{negative mass}} \Big]$$

$$= \frac{-\rho 18 \pi r^4}{\rho (240 r^3 - 6\pi r^3)}$$

$$= \frac{-18\pi r}{240 - 6\pi}$$

$$\bar{z} = -0.256 r$$

The moments for the solid prism are to be found.

$$I_{x_{sp}} = \int r^2 dm \qquad r^2 = y^2 + z^2 \quad dm = \rho\, dx\, dy\, dz$$

$$= \rho \iiint (y^2+z^2)\, dx\, dy\, dz$$

$$= Y + Z$$

$$= 320\rho r^5 + 2000\rho r^5$$

$$I_{x_{sp}} = 2320\rho r^5$$

The // axis theorem is used to find the moment at the center of mass.

$$I_{x_{cm_{sp}}} = I_{x_{sp}} + M(\bar{z})^2$$

$$= 2320\rho r^5 + 240\rho r^3(-.26)^2$$

$$= 2320\rho r^5 + 16.22\rho r^5$$

$$I_{x_{cm_{sp}}} = 2336.22\rho r^5$$

$$I_{x_{cm_{sc}}} = I_{x_{sc}} + \rho\pi r^2(6r)(3r+.26r)^2$$

$$= \frac{\rho\pi r^4(6r)}{2} + \rho 6\pi r^3(10.63r^2)$$

$$= \rho 3\pi r^5[1 + 2(10.63)]$$

$$= \rho 3\pi r^5[22.26]$$

$$I_{x_{cm_{sc}}} = 209.80$$

$$I_x = I_{x_{cm_{sp}}} - I_{x_{cm_{sc}}}$$

$$= (2336.22 - 209.80)\rho r^5$$

$$\boxed{I_x = 2126\rho\, cm^5 \quad \{r = 1cm\}}$$

$$I_{ysp} = \int r^2 dm \qquad r^2 = x^2 + z^2, \; dm = \rho\, dx\, dy\, dz$$

$$= \rho \iiint (x^2 + z^2)\, dm$$

$$= X + Z$$

$$= 720\rho r^5 + 2000\rho r^5$$

$$I_{ysp} = 2720\rho r^5$$

Using the // axis theorem

$$I_{ycmsp} = I_{ysp} + M(\bar{z})^2$$

$$= 2720\rho r^5 + 240 r^3 (.26r)^2$$

$$= 2720\rho r^5 + 16.22\rho r^5$$

$$I_{ycmsp} = 2736.22\rho r^5$$

$$I_{ycmsc} = I_{ysc} + M(3r + .26r)^2$$

$$= \frac{\rho \pi r^2 h}{4} + \frac{\rho \pi r^2 h^3}{12} + \rho \pi r^2 h (3.26r)^2$$

$$= \rho \pi r^2 \left(\frac{6r^3}{4} + \frac{(6r)^3}{12} + 6r(3.26r)^2 \right)$$

$$= \rho \pi r^5 \left(\frac{3}{2} + 18 + 63.77 \right)$$

$$I_{ycmsc} = 261.60\rho r^5$$

$$I_y = I_{ycmsp} - I_{ycmsc}$$

$$= (2736.22 - 261.60)\rho r^5$$

$$\boxed{I_y = 2474\rho cm^5}$$

$$I_{zsp} = \int r^2 dm \qquad r^2 = x^2 + y^2, \; dm = \rho\, dx\, dy\, dz$$

$$= \rho \iiint (x^2+y^2)\, dm$$

$$= X + Y$$

$$= 720\rho r^5 + 320\rho r^5$$

$$I_{zsp} = 1040\rho r^5$$

Using the // axis theorem

$$I_{zcmsp} = I_{zsp} + M(\bar{z})^2$$

$$= 1040\rho r^5 + 16.22\rho r^5$$

$$I_{zcmsp} = 1056.22\rho r^5$$

The cylinder c.m. is along the z-axis

$$I_z = I_{zcmsp} - I_{zsc}$$

$$= 1056.22\rho r^5 - \rho\pi r^2\left(\frac{r^2 h}{4} + \frac{h^3}{12}\right)$$

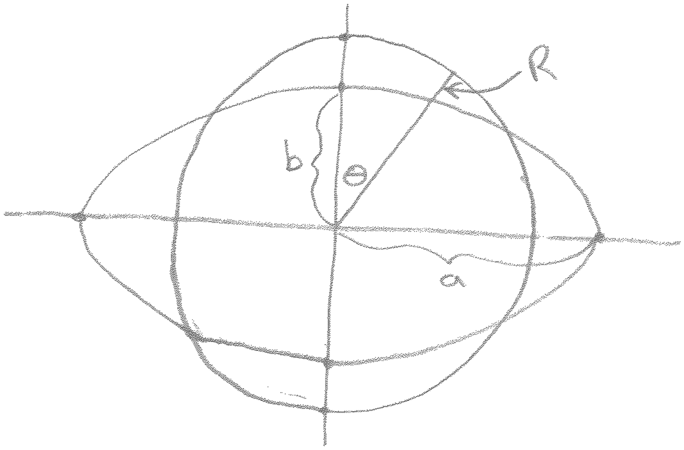
$$= 1056.22\rho r^5 - \rho\pi r^5\left(\frac{6}{4} + 18\right)$$

$$= 1056.22\rho r^5 - 61.26\rho r^5$$

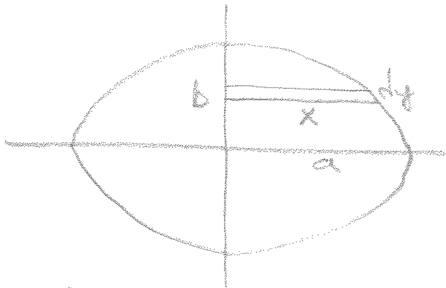
$$\boxed{I_z = 995\rho cm^5}$$

6-17

a)



Volume of ellipse



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$x^2 = a^2\left(1 - \frac{y^2}{b^2}\right)$$

$$dV = \pi x^2 dy$$

$$V = 2\pi a^2 \int_0^b \left(1 - \frac{y^2}{b^2}\right) dy$$

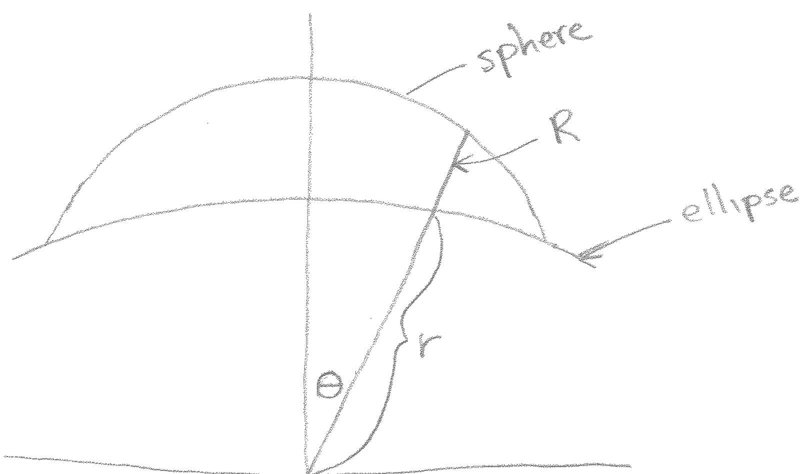
$$= 2\pi a^2 \left(b - \frac{b}{3}\right)$$

$$= \frac{4}{3}\pi a^2 b \qquad b = a(1-n)$$

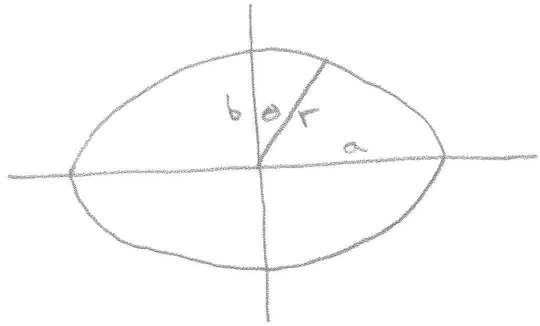
$$\boxed{V = \frac{4}{3}\pi a^3 (1-n)}$$

$\therefore\ R^3 = a^3(1-n)$ {equivalent spherical volume of radius R}

$R = a(1-n)^{1/3}$.



Find expression for r to first order m.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad y=r\cos\theta$$

$$x=r\sin\theta \qquad b=a(1-m)$$

$$b^2x^2+a^2y^2=a^2b^2$$

$$a^2(1-m)^2r^2\sin^2\theta+a^2r^2\cos^2\theta=a^4(1-m)^2$$

$$r^2\left[\sin^2\theta+\frac{\cos^2\theta}{(1-m)^2}\right]=a^2$$

$$r^2\left[(1-\cos^2\theta)+(1-m)^{-2}\cos^2\theta\right]=a^2$$

$$r^2\left[1-\cos^2\theta+(1+2m)\cos^2\theta\right]=a^2 \quad \{\text{first order } m\}$$

$$r^2\left[1-\cos^2\theta+\cos^2\theta+2m\cos^2\theta\right]=a^2$$

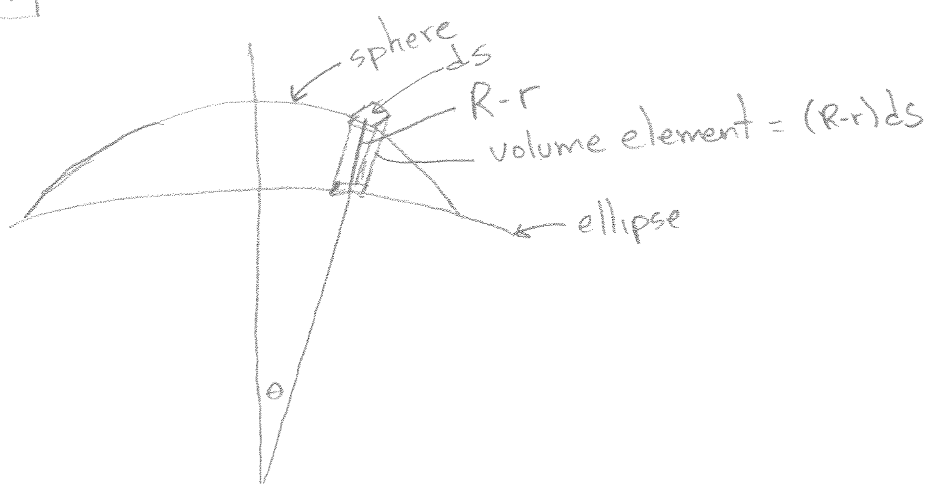
$$r^2\left[1+2m\cos^2\theta\right]=a^2$$

$$r^2=\frac{a^2}{1+2m\cos^2\theta}$$

$$r=\frac{a}{(1+2m\cos^2\theta)^{1/2}}$$

$$r=a(1+2m\cos^2\theta)^{-1/2}$$

$$\boxed{r=a(1-m\cos^2\theta)}$$



$$R - r = a(1-n)^{1/3} - a(1 - n\cos^2\theta)$$

$$\cong a(1 - \tfrac{1}{3}n) - a + an\cos^2\theta$$

$$= a - \tfrac{1}{3}an - a + an\cos^2\theta$$

$$= -\tfrac{1}{3}an + an\cos^2\theta$$

$$dm = \rho\, dV$$

$$= \rho(R-r)\, dS$$

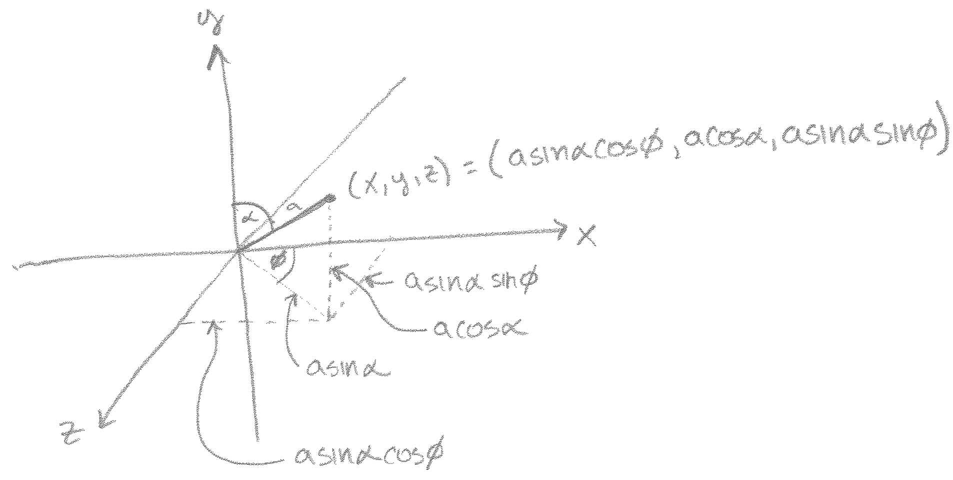
$$\frac{dm}{dS} = \rho\left(-\tfrac{1}{3}an + an\cos^2\theta\right)$$

$$\sigma = \tfrac{1}{3}na\rho(3\cos^2\theta - 1)$$

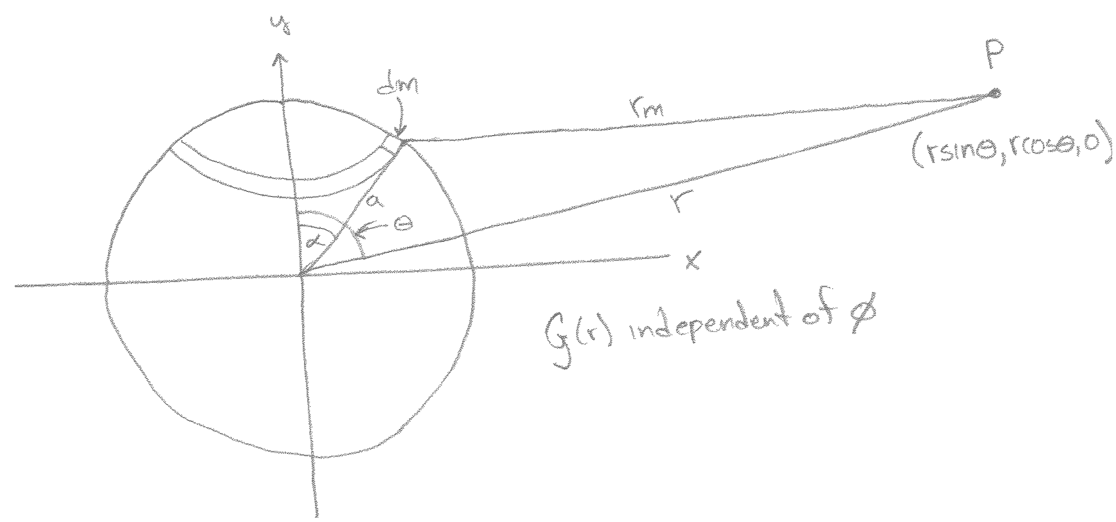
$\therefore$ the surface density to create the ellipsoid is $-\sigma$

$$\boxed{\sigma = \tfrac{1}{3}na\rho(1 - 3\cos^2\theta)}$$

b)



$$da = (a\sin\alpha\, d\phi)(a\, d\alpha)$$



$$\vec{r}_a = a\sin\alpha\cos\phi\,\hat{i} + a\cos\alpha\,\hat{j} + a\sin\alpha\sin\phi\,\hat{k}$$

$$\vec{r} = r\sin\theta\,\hat{i} + r\cos\theta\,\hat{j}$$

$$|\vec{r}_m| = |\vec{r}_a - \vec{r}|$$

$$= \left[(a\sin\alpha\cos\phi - r\sin\theta)^2 + (a\cos\alpha - r\cos\theta)^2 + a^2\sin^2\alpha\sin^2\phi\right]^{1/2}$$

$$= \left[a^2\sin^2\alpha\cos^2\phi + r^2\sin^2\theta - 2ar\sin\alpha\cos\phi\sin\theta + a^2\cos^2\alpha + r^2\cos^2\theta - 2ar\cos\alpha\cos\theta + a^2\sin^2\alpha\sin^2\phi\right]^{1/2}$$

$$= \left[r^2(\sin^2\theta + \cos^2\theta) + a^2\sin^2\alpha(\cos^2\phi + \sin^2\phi) + a^2\cos^2\alpha - 2ar(\sin\alpha\cos\phi\sin\theta + \cos\alpha\cos\theta)\right]^{1/2}$$

$$= \left[r^2 + a^2 - 2ar(\sin\alpha\cos\phi\sin\theta + \cos\alpha\cos\theta)\right]^{1/2}$$

$$= \left[r^2 + a^2 - 2ar\, f(\alpha,\phi,\theta)\right]^{1/2}$$

$$(1+b)^n = 1 + nb + \frac{n(n-1)}{2!}b^2 + \frac{n(n-1)(n-2)}{3!}b^3 + \cdots$$

$$\frac{1}{|\vec{r}_m|} = \frac{1}{r}\left[1 + \frac{a^2}{r^2} - \frac{2a}{r}f(\alpha,\phi,\theta)\right]^{-1/2}$$

$$b = \frac{a^2}{r^2} - \frac{2a}{r}f(\alpha,\phi,\theta), \quad n = -1/2$$

$$\frac{1}{|\vec{r}_m|} = \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{a^2}{r^2} - \frac{2a}{r}f(\alpha,\phi,\theta)\right) + \frac{(-\frac{1}{2})(-\frac{3}{2})}{2}\left(\frac{a^2}{r^2} - \frac{2a}{r}f(\alpha,\phi,\theta)\right)^2 + \cdots\right.$$

$$\frac{1}{|\vec{r}_m|} \doteq \frac{1}{r}\left[1 - \frac{a^2}{2r^2} + \frac{a}{r}f(\alpha\phi,\theta) + \frac{3}{8}\left[\frac{a^4}{r^4} + \frac{4a^2}{r^2}f^2(\alpha,\phi,\theta) - \frac{4a^3}{r^3}f(\alpha,\phi,\theta)\right]\right]$$

$$dm = \sigma\, da \qquad \sigma = \frac{1}{3}na\rho(1-3\cos^2\alpha) \quad da = a\sin\alpha\, d\phi\, a\, d\alpha$$

$$= \frac{1}{3}na^3\rho\sin\alpha(1-3\cos^2\alpha)\,d\alpha\, d\phi$$

$$\delta G = G_{surf} = \int \frac{G\,dm}{|\vec{r}_m|}$$

$$\doteq \frac{Gma^3\rho}{3r}\int \sin\alpha(1-3\cos^2\alpha)\left[1 - \frac{a^2}{2r^2} + \frac{a}{r}f(\alpha,\phi,\theta) + \frac{3}{8}\left[\frac{4a^2}{r^2}f^2(\alpha,\phi,\theta)\right]\right] \text{ to order } \frac{a^3}{r^3}$$

$$\doteq \frac{2Gma^3\rho}{3r}\int_0^\pi\int_0^\pi (\sin\alpha - 3\cos^2\alpha\sin\alpha)\left[1 - \frac{a^2}{2r^2} + \frac{a}{r}f(\alpha,\phi,\theta) + \frac{3}{2}\frac{a^2}{r^2}f^2(\alpha,\phi,\theta)\right]d\alpha\, d\phi$$

i) $\displaystyle\int_0^\pi(\sin\alpha - 3\cos^2\alpha\sin\alpha)\,d\alpha = \int_0^\pi \sin\alpha\, d\alpha - 3\int_0^\pi \cos^2\alpha\sin\alpha\, d\alpha$ $\quad u = \cos\alpha \rightarrow du = -\sin\alpha\, d\alpha$

$$= -\cos\alpha\Big|_0^\pi + \cos^3\alpha\Big|_0^\pi$$

$$= -(-1-1) + (-1-1)$$

$$= 0$$

ii) $\displaystyle\int_0^\pi(\sin\alpha - 3\cos^2\alpha\sin\alpha)f(\alpha,\phi,\theta)\,d\alpha = \int_0^\pi(\sin\alpha - 3\cos^2\alpha\sin\alpha)(\sin\alpha\cos\phi\sin\theta + \cos\alpha\cos\theta)\,d\alpha$

$$= \int_0^\pi(\sin^2\alpha\cos\phi\sin\theta + \sin\alpha\cos\alpha\cos\theta - 3\cos^2\alpha\sin^2\alpha\cos\phi\sin\theta - 3\cos^3\alpha\sin\alpha\cos\theta)\,d\alpha$$

$$= \int_0^\pi \frac{1}{2}(1+\cos 2\alpha)\,d\alpha\cos\phi\sin\theta + \int_0^\pi \sin\alpha\cos\alpha\, d\alpha\cos\theta - 3\int_0^\pi\cos^2\alpha\sin^2\alpha\, d\alpha\cos\phi\sin\theta -$$

$$3\int_0^\pi \cos^3\alpha\sin\alpha\, d\alpha\cos\theta$$

$$= \left(\frac{\pi}{2} + \frac{1}{4}\sin 2\alpha\Big|_0^\pi\right)\cos\phi\sin\theta + \frac{\sin^2\alpha}{2}\Big|_0^\pi\cos\theta - 3\int_0^\pi\frac{1}{2}(1+\cos 2\alpha)\frac{1}{2}(1-\cos 2\alpha)\,d\alpha\cos\phi\sin\theta +$$

$$\cos^4\alpha\Big|_0^\pi\cos\theta$$

$$= \frac{\pi}{2}\cos\phi\sin\theta + 0 - \frac{3}{4}\int_0^\pi(1-\cos^2 2\alpha)\,d\alpha\cos\phi\sin\theta + 0$$

$$= \frac{\pi}{2}\cos\phi\sin\theta - \frac{3}{4}\int_0^\pi\left(1 - \frac{1}{2}(1+\cos 4\alpha)\right)d\alpha\cos\phi\sin\theta$$

$$= \frac{\pi}{2}\cos\phi\sin\theta - \frac{3}{4}\int_0^\pi\left(\frac{1}{2} - \frac{1}{2}\cos 4\alpha\right)d\alpha\cos\phi\sin\theta$$

$$= \frac{\pi}{2}\cos\phi\sin\theta - \frac{3}{4}\left(\frac{\pi}{2} - \frac{1}{8}\sin 4\alpha\Big|_0^{\pi}\right)\cos\phi\sin\theta$$

$$= \frac{\pi}{2}\cos\phi\sin\theta - \frac{3\pi}{8}\cos\phi\sin\theta$$

$$= \left(\frac{\pi}{2} - \frac{3\pi}{8}\right)\cos\phi\sin\theta$$

$$= \frac{\pi}{8}\cos\phi\sin\theta$$

iii) $\int_0^{\pi}(\sin\alpha - 3\cos^2\alpha\sin\alpha)\, f^2(\alpha,\phi,\theta)\,d\alpha = \int_0^{\pi}(\sin\alpha - 3\cos^2\alpha\sin\alpha)(\sin\alpha\cos\phi\sin\theta + \cos\alpha\cos\theta)^2\,d\alpha$

$$= \int_0^{\pi}(\sin\alpha - 3\cos^2\alpha\sin\alpha)\left(\sin^2\alpha\cos^2\phi\sin^2\theta + \cos^2\alpha\cos^2\theta + 2\sin\alpha\cos\alpha\cos\phi\sin\theta\cos\theta\right)d\alpha$$

$$= \int_0^{\pi}\sin^3\alpha\cos^2\phi\sin^2\theta\,d\alpha + \int_0^{\pi}\cos^2\alpha\sin\alpha\cos^2\theta\,d\alpha + 2\int_0^{\pi}\sin^2\alpha\cos\alpha\cos\phi\sin\theta\cos\theta\,d\alpha$$

$$-3\int_0^{\pi}\sin^3\alpha\cos^2\alpha\cos^2\phi\sin^2\theta\,d\alpha - 3\int_0^{\pi}\cos^4\alpha\sin\alpha\cos^2\theta\,d\alpha - 6\int_0^{\pi}\cos^3\alpha\sin^2\alpha\cos\phi\sin\theta\cos\theta\,d\alpha$$

$$= \int_0^{\pi}\sin\alpha(1-\cos^2\alpha)\,d\alpha\cos^2\phi\sin^2\theta - \frac{\cos^3\alpha}{3}\Big|_0^{\pi}\cos^2\theta + 2\,\frac{\sin^3\alpha}{3}\Big|_0^{\pi}\cos\phi\sin\theta\cos\theta$$

$$-3\int_0^{\pi}\sin\alpha(1-\cos^2\alpha)\cos^2\alpha\,d\alpha\cos^2\phi\sin^2\theta + 3\,\frac{\cos^5\alpha}{5}\Big|_0^{\pi}\cos^2\theta - 6\int_0^{\pi}\cos\alpha(1-\sin^2\alpha)\sin^2\alpha\,d\alpha\cos\phi\sin\theta\cos\theta$$

$$= \int_0^{\pi}[\sin\alpha - \cos^2\alpha\sin\alpha]\,d\alpha\cos^2\phi\sin^2\theta - \left(-\frac{1}{3} - \frac{1}{3}\right)\cos^2\theta + 0 - 3\int_0^{\pi}[\cos^2\alpha\sin\alpha - \cos^4\alpha\sin\alpha]\,d\alpha\cos^2\phi\sin^2\theta$$

$$+\frac{3}{5}(-1-1)\cos^2\theta - 6\int_0^{\pi}[\sin^2\alpha\cos\alpha - \sin^4\alpha\cos\alpha]\,d\alpha\cos\phi\sin\theta\cos\theta$$

$$= \left[-\cos\alpha\Big|_0^{\pi} + \frac{\cos^3\alpha}{3}\Big|_0^{\pi}\right]\cos^2\phi\sin^2\theta + \frac{2}{3}\cos^2\theta - 3\left[\frac{-\cos^3\alpha}{3}\Big|_0^{\pi} + \frac{\cos^5\alpha}{5}\Big|_0^{\pi}\right]\cos^2\phi\sin^2\theta$$

$$-\frac{6}{5}\cos^2\theta - 6\left[\frac{\sin^3\alpha}{3}\Big|_0^{\pi} - \frac{\sin^5\alpha}{5}\Big|_0^{\pi}\right]\cos\phi\sin\theta\cos\theta$$

$$= \left[-(-1-1) + \left(-\frac{1}{3} - \frac{1}{3}\right)\right]\cos^2\phi\sin^2\theta + \frac{2}{3}\cos^2\theta - 3\left[\frac{-(-1-1)}{3} + \left(-\frac{1}{5} - \frac{1}{5}\right)\right]\cos^2\phi\sin^2\theta$$

$$-\frac{6}{5}\cos^2\theta - 0$$

$$= \left[2 - \frac{2}{3}\right]\cos^2\phi\sin^2\theta + \frac{2}{3}\cos^2\theta - 3\left[\frac{2}{3} - \frac{2}{5}\right]\cos^2\phi\sin^2\theta - \frac{6}{5}\cos^2\theta$$

$$= \frac{4}{3}\cos^2\phi\sin^2\theta + \frac{2}{3}\cos^2\theta - 3\left[\frac{4}{15}\right]\cos^2\phi\sin^2\theta - \frac{6}{5}\cos^2\theta$$

$$= \left[\frac{4}{3} - \frac{12}{15}\right]\cos^2\phi\sin^2\theta + \left[\frac{2}{3} - \frac{6}{5}\right]\cos^2\theta$$

$$= \frac{8}{15}\cos^2\phi\sin^2\theta - \frac{8}{15}\cos^2\theta$$

$$G_{surf} \doteq \frac{2Gma^3 l}{3r} \int_0^{\pi} \left[\frac{a}{r}\frac{\pi}{8}\cos\phi \sin\theta + \frac{3}{2}\frac{a^2}{r^2}\left[\frac{8}{15}\cos^2\phi \sin^2\theta - \frac{8}{15}\cos^2\theta\right]\right] d\phi$$

$$\int_0^{\pi} \cos\phi \, d\phi \sin\theta = \sin\phi\Big|_0^{\pi} \sin\theta = 0$$

$$\int_0^{\pi} \cos^2\phi \, d\phi \sin^2\theta = \frac{1}{2}\int_0^{\pi}(1+\cos 2\phi)\,d\phi \sin^2\theta = \left[\frac{\pi}{2} + \frac{1}{2}\sin 2\phi\Big|_0^{\pi}\right]\sin^2\theta = \frac{\pi}{2}\sin^2\theta$$

$$G_{surf} \doteq \frac{2Gma^3 l}{3r}\left[\frac{3}{2}\frac{a^2}{r^2}\left[\frac{8}{15}\frac{\pi}{2}\sin^2\theta - \frac{8}{15}\cos^2\theta\,\pi\right]\right]$$

$$\doteq \frac{4\pi a^3 Gma^2}{3r^3}\left[\frac{1}{5}\sin^2\theta - \frac{2}{5}\cos^2\theta\right]$$

$$\doteq \frac{MGma^2}{r^3}\left[\frac{1}{5}(1-\cos^2\theta) - \frac{2}{5}\cos^2\theta\right] \quad \{a \doteq R, \text{ radius of shell}\}$$

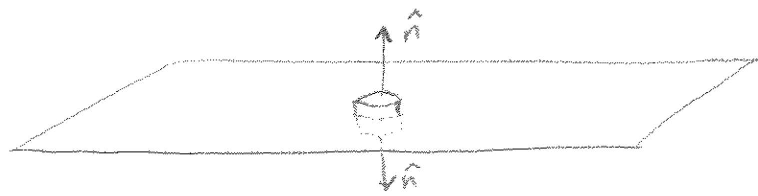
$$\doteq \frac{MGma^2}{r^3}\left[\frac{1}{5} + \left(\frac{-1}{5} - \frac{2}{5}\right)\cos^2\theta\right]$$

$$\doteq \frac{MGma^2}{r^3}\left[\frac{1}{5} - \frac{3}{5}\cos^2\theta\right]$$

$$G_{surf} \doteq \frac{1}{5} m \frac{MGa^2}{r^3}\left[1 - 3\cos^2\theta\right]$$

$$\boxed{\delta G \doteq \frac{1}{5} m \frac{MGa^2}{r^3}\left[1 - 3\cos^2\theta\right]}$$

6-19



a) $\iint \hat{n} \cdot \vec{g}\, ds = -4\pi G \iint dm$

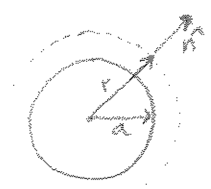
$$= -4\pi G \iint \sigma\, ds$$

$$g \iint ds = -4\pi G \sigma \iint ds$$

$$g(2A) = -4\pi G \sigma A$$

$$g = -2\pi G \sigma$$

b)



For $r \geq a$ the field behaves as if all the mass were at the center of the shell. At $r = a$,

$$\iint \hat{n} \cdot \hat{g}\, ds = -4\pi G \iint dm$$

$$= -4\pi G \sigma \iint ds$$

$$g \iint ds = -4\pi G \sigma\, 4\pi a^2$$

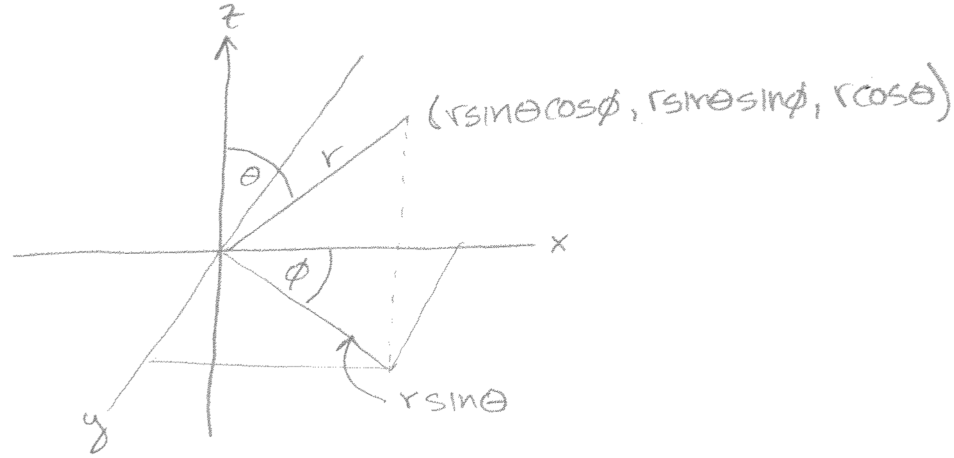
$$g\, 4\pi a^2 = -4\pi G \sigma\, 4\pi a^2$$

$$g = -4\pi G \sigma$$

Half of the field comes from the adjacent matter and half comes from the distant matter of the sheet.

6-21

a)



$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

In spherical coordinates

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial \phi^2}$$

For azimuthal symmetry, $\frac{\partial f}{\partial \phi} = 0$, and Laplace's Equation is $\nabla^2 f = 0$ giving

1) $$\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) = 0$$

$$f = \delta G = \frac{1}{5}\eta\, MGa^2 \frac{(1-3\cos^2\theta)}{r^3}$$

$$= K\frac{(1-3\cos^2\theta)}{r^3}, \quad K = \frac{1}{5}\eta\, MGa^2$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) = K\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\left(\frac{1-3\cos^2\theta}{r^3}\right)\right) = K\frac{\partial}{\partial r}\left(\frac{r^2(-3)(1-3\cos^2\theta)}{r^4}\right) = K\frac{\partial}{\partial r}\left(\frac{-3(1-3\cos^2\theta)}{r^2}\right)$$

$$= \frac{6K(1-3\cos^2\theta)}{r^3}$$

$$\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial f}{\partial \theta}\right) = \frac{K}{\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial}{\partial \theta}\left(\frac{(1-3\cos^2\theta)}{r^3}\right)\right) = \frac{K}{r^3 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta(6\cos\theta\sin\theta)\right)$$

$$= \frac{6K}{r^3 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin^2\theta\cos\theta\right) = \frac{6K}{r^3 \sin\theta}\left(-\sin^3\theta + 2\sin\theta\cos^2\theta\right)$$

$$= \frac{6K}{r^3}\left(2\cos^2\theta - \sin^2\theta\right) = \frac{6K}{r^3}\left(2\cos^2\theta - (1-\cos^2\theta)\right)$$

$$= \frac{6K}{r^3}\left(-1 + 3\cos^2\theta\right)$$

$$= -\frac{6K}{r^3}\left(1 - 3\cos^2\theta\right)$$

Substituting the partial derivative expressions into Eq. 1 gives

$$\frac{6K}{r^3}(1-3\cos^2\theta) - \frac{6K}{r^3}(1-3\cos^2\theta) = 0$$

Therefore, δG satisfies Laplace's Equation $\nabla^2 f = 0$

b) Poisson's Equation:

$$\nabla^2 G = -4\pi G \rho$$

$$\rho = f(r)(1-3\cos^2\theta)$$

Given a solution of $G = h(r)(1-3\cos^2\theta)$ find $\nabla^2 G$ for azimuthal symmetry

$$\nabla^2 G = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial G}{\partial\theta}\right)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right)(1-3\cos^2\theta) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(h\sin\theta\,(6\cos\theta\sin\theta)\right)$$

$$= \frac{1}{r^2}\left(r^2\frac{\partial^2 h}{\partial r^2} + 2r\frac{\partial h}{\partial r}\right)(1-3\cos^2\theta) + \frac{6h}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin^2\theta\cos\theta\right)$$

$$= \left[\frac{\partial^2 h}{\partial r^2} + \frac{2}{r}\frac{\partial h}{\partial r}\right](1-3\cos^2\theta) + \frac{6h}{r^2\sin\theta}\left[-\sin^3\theta + 2\sin\theta\cos^2\theta\right]$$

$$= \left[\frac{\partial^2 h}{\partial r^2} + \frac{2}{r}\frac{\partial h}{\partial r}\right](1-3\cos^2\theta) + \frac{6h}{r^2}\left[2\cos^2\theta - \sin^2\theta\right]$$

$$= \left[\frac{\partial^2 h}{\partial r^2} + \frac{2}{r}\frac{\partial h}{\partial r}\right](1-3\cos^2\theta) + \frac{6h}{r^2}\left[2\cos^2\theta - (1-\cos^2\theta)\right]$$

$$= \left[\frac{\partial^2 h}{\partial r^2} + \frac{2}{r}\frac{\partial h}{\partial r}\right](1-3\cos^2\theta) + \frac{6h}{r^2}\left[-1+3\cos^2\theta\right]$$

$$\nabla^2 G = \left[\frac{\partial^2 h}{\partial r^2} + \frac{2}{r}\frac{\partial h}{\partial r} - \frac{6h}{r^2}\right](1-3\cos^2\theta)$$

$$\nabla^2 G = -4\pi G\rho$$

$$\left[\frac{\partial^2 h}{\partial r^2} + \frac{2}{r}\frac{\partial h}{\partial r} - \frac{6h}{r^2}\right](1-3\cos^2\theta) = -4\pi G f(1-3\cos^2\theta)$$

$$\boxed{\frac{d^2 h}{dr^2} + \frac{2}{r}\frac{dh}{dr} - \frac{6h}{r^2} = -4\pi G f}$$

c) $h = r^{-3}$, $f = 0$

$$\frac{d^2}{dr^2}(r^{-3}) + \frac{2}{r}\frac{d}{dr}(r^{-3}) - \frac{6}{r^2}(r^{-3}) =$$

$$\frac{d}{dr}(-3r^{-4}) + \frac{2}{r}(-3r^{-4}) - \frac{6}{r^5} =$$

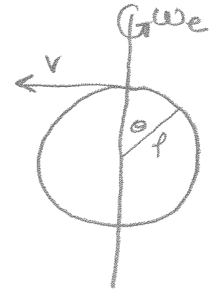
$$\frac{12}{r^5} - \frac{6}{r^5} - \frac{6}{r^5} = 0$$

$\therefore$ when $h(r) = r^{-3}$, Laplace's Equation will be satified ($f=0$).

Conclusion:

when $G = Kh(r)(1-3\cos^2\theta)$ where $h(r) = r^{-3}$, Laplace's Equation will be satified. therefore G is valid everywhere outside of the mass distribution ($\rho = f(r)(1-3\cos^2\theta)$ and $f(r) = 0$).

7-9



$$\omega_e = \frac{2\pi \text{ rad}}{24 \text{ hr}} \cdot \frac{\text{hr}}{3600 \text{ s}} = 7.27 \times 10^{-5} \frac{1}{s}$$

The equation of motion for a rotating coordinate system is

$$m \frac{d^{*2}\vec{r}}{dt^2} = \vec{F} + m(\vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{r})) - 2m\vec{\omega} \times \frac{d^*\vec{r}}{dt}$$

First, the Coriolis acceleration is found. Using spherical coordinates

$$\frac{d^*\vec{r}}{dt} = V_r \hat{r} + V_\theta \hat{\theta} + V_\phi \hat{\phi}$$

$V_r = V_\phi = 0$ {relative to moving coordinate system}

$$V_\theta = 500 \frac{mi}{hr} \times \frac{5280 \text{ ft}}{mi} \times \frac{12 \text{ in}}{ft} \times \frac{.0254 \text{ m}}{in} \times \frac{hr}{3600 \text{ s}} = 223.5 \frac{m}{s}$$

$$\vec{a}_{cor} = -2\vec{\omega}_e \times \vec{V}_\theta$$

$$|a_{cor}| = 2\omega_e V_\theta \quad \{\vec{\omega}_e \perp \vec{V}_\theta\}$$

$$= 2(7.27 \times 10^{-5} \tfrac{1}{s})(223.5 \tfrac{m}{s})$$

$$|a_{cor}| = 3.25 \times 10^{-2} \frac{m}{s^2}$$

Since the moving plumb bob is supported by a string no compressive forces exist in the string. Therefore, a centrifugal force displaces the plumb bob.

$$\vec{a}_{cen} = \vec{\omega}_p \times (\vec{\omega}_p \times \vec{r}_e) \qquad \omega_p = \frac{V_p}{R_e}, \; r_e = R_e \text{ at the pole}$$

$$= 6357 \times 10^3 \text{ m}$$

$$= \omega_p^2 r_e \hat{r}$$

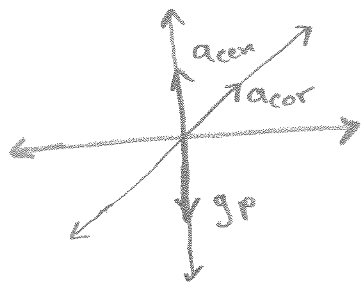
$$= \frac{V_p^2}{R_e} \qquad = \frac{223.5 \frac{m}{s}}{6357 \times 10^3 \text{ m}} = 3.52 \times 10^{-5} \frac{1}{s}$$

$$= 7.86 \times 10^{-3} \frac{m}{s^2} \hat{r}$$

At the pole the gravitational acceleration is

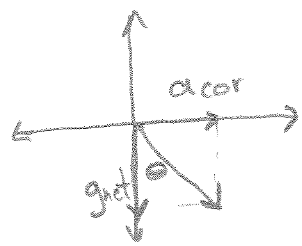
$$g_p = 9.87 \frac{m}{s^2}$$

The accelerations present on the moving plumb bob are



$$g_{net} = a_{cen} - g_p$$

$$= 9.86 \frac{m}{s^2}$$



$$\theta = \tan^{-1}\left(\frac{a_{cor}}{g_{net}}\right)$$

$$= \tan^{-1}\left(\frac{3.25 \times 10^{-2}}{9.86}\right)$$

$$\theta = 3.30 \times 10^{-3} \text{rad}$$

The plumb bob at rest situated at the pole does not experience a Coriolis force because $\frac{d^* \vec{r}}{dt} = 0$

Therefore, the angle between the moving plumb bob and the freely hanging one situatated at the pole is

$$\boxed{\theta = 3.30 \times 10^{-3} \text{rad}}$$

Note: answer in text is 5.25×10^{-4} rad

7-19

Part 1.

$$'V' = \frac{m(M_1+M_2)G}{a}\left\{\frac{\xi_2}{[(\xi-\xi_1)^2+\eta^2]^{1/2}} - \frac{\xi_1}{[(\xi-\xi_2)^2+\eta^2]^{1/2}} - \frac{1}{2}(\xi^2-\eta^2)\right\}$$

$$\xi = \frac{x}{a}, \quad \xi_1 = \frac{M_2}{M_1+M_2}, \quad \xi_2 = -\frac{M_1}{M_1+M_2} = \xi_1 - 1$$

$$\eta = \frac{y}{a}$$

$$\frac{\partial 'V'}{\partial \xi} = \frac{m(M_1+M_2)G}{a}\left\{\frac{-1}{2}\xi_2[(\xi-\xi_1)^2+\eta^2]^{-3/2}2(\xi-\xi_1) - \frac{-1}{2}\xi_1[(\xi-\xi_2)^2+\eta^2]^{-3/2}2(\xi-\xi_2) - \xi\right\}$$

$$= \frac{m(M_1+M_2)G}{a}\left\{-\xi_2(\xi-\xi_1)[(\xi-\xi_1)^2+\eta^2]^{-3/2} + \xi_1(\xi-\xi_2)[(\xi-\xi_2)^2+\eta^2]^{-3/2} - \xi\right\}$$

$$\frac{\partial^2 'V'}{\partial \xi^2} = \frac{m(M_1+M_2)G}{a}\left\{-\xi_2(\xi-\xi_1)\left(-\frac{3}{2}\right)[(\xi-\xi_1)^2+\eta^2]^{-5/2}2(\xi-\xi_1) + [(\xi-\xi_1)^2+\eta^2]^{-3/2}(-\xi_2) + \right.$$
$$\left. \xi_1(\xi-\xi_2)\left(-\frac{3}{2}\right)[(\xi-\xi_2)^2+\eta^2]^{-5/2}2(\xi-\xi_2) + [(\xi-\xi_2)^2-\eta^2]^{-3/2}(\xi_1) - 1\right\}$$

$$\frac{\partial^2 'V'}{\partial \xi^2} = \frac{m(M_1+M_2)G}{a}\left\{3\xi_2(\xi-\xi_1)^2[(\xi-\xi_1)^2+\eta^2]^{-5/2} - \xi_2[(\xi-\xi_1)^2+\eta^2]^{-3/2} - \right.$$
$$\left. 3\xi_1(\xi-\xi_2)^2[(\xi-\xi_2)^2+\eta^2]^{-5/2} + \xi_1[(\xi-\xi_2)^2+\eta^2]^{-3/2} - 1\right\}$$

$$\frac{\partial 'V'}{\partial \eta} = \frac{m(M_1+M_2)G}{a}\left\{-\frac{1}{2}\xi_2[(\xi-\xi_1)^2+\eta^2]^{-3/2}2\eta - \frac{-1}{2}\xi_1[(\xi-\xi_2)^2+\eta^2]^{-3/2}2\eta + \eta\right\}$$

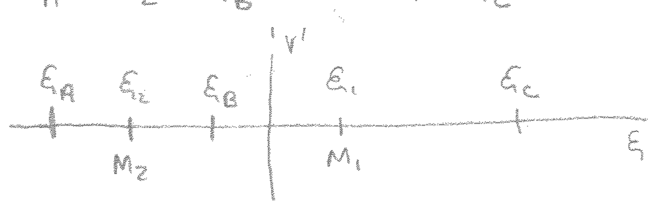
$$= \frac{m(M_1+M_2)G}{a}\left\{-\xi_2\eta[(\xi-\xi_1)^2+\eta^2]^{-3/2} + \xi_1\eta[(\xi-\xi_2)^2+\eta^2]^{-3/2} + \eta\right\}$$

$$\frac{\partial^2 'V'}{\partial \eta^2} = \frac{m(M_1+M_2)G}{a}\left\{-\xi_2\eta\left(-\frac{3}{2}\right)[(\xi-\xi_1)^2+\eta^2]^{-5/2}2\eta - \xi_2[(\xi-\xi_1)^2+\eta^2]^{-3/2} + \right.$$
$$\left. \xi_1\eta\left(-\frac{3}{2}\right)[(\xi-\xi_2)^2+\eta^2]^{-5/2}2\eta + \xi_1[(\xi-\xi_2)^2+\eta^2]^{-3/2} + 1\right\}$$

$$\frac{\partial^2 'V'}{\partial \eta^2} = \frac{m(M_1+M_2)G}{a}\left\{3\xi_2\eta^2[(\xi-\xi_1)^2+\eta^2]^{-5/2} - \xi_2[(\xi-\xi_1)^2+\eta^2]^{-3/2} - \right.$$
$$\left. 3\xi_1\eta^2[(\xi-\xi_2)^2+\eta^2]^{-5/2} + \xi_1[(\xi-\xi_2)^2+\eta^2]^{-3/2} + 1\right\}$$

$$\frac{\partial}{\partial \xi}\frac{\partial 'V'}{\partial \eta} = \frac{m(M_1+M_2)G}{a}\left\{-\xi_2\eta\left(-\frac{3}{2}\right)[(\xi-\xi_1)^2+\eta^2]^{-5/2}2(\xi-\xi_1) + \xi_1\eta\left(-\frac{3}{2}\right)[(\xi-\xi_2)^2+\eta^2]^{-5/2}2(\xi-\xi_2)\right\}$$

$$\xi_A < \xi_2 < \xi_B < 0 < \xi_1 < \xi_C$$



Let $K = \frac{m(M_1+M_2)G}{a}$, $\eta = 0$, $\xi_L = \xi_{A,B,C}$, $\xi_2 = \xi_1 - 1$

$$\frac{\partial^2 V'}{\partial \xi^2} = K\left\{ \frac{3\xi_2(\xi_L-\xi_1)^2}{|\xi_L-\xi_1|^5} - \frac{3\xi_1(\xi_L-\xi_2)^2}{|\xi_L-\xi_2|^5} - \frac{\xi_2}{|\xi_L-\xi_1|^3} + \frac{\xi_1}{|\xi_L-\xi_2|^3} - 1 \right\}$$

$$\frac{(\xi_L-\xi_1)^2}{|\xi_L-\xi_1|^5} = \frac{\left[(\xi_L-\xi_1)^2\right]^1}{\left[(\xi_L-\xi_1)^2\right]^{5/2}} = \left[(\xi_L-\xi_1)^2\right]^{-3/2} = \frac{1}{|\xi_L-\xi_1|^3}$$

$$\frac{\partial^2 V'}{\partial \xi^2} = K\left\{ \frac{3\xi_2}{|\xi_L-\xi_1|^3} - \frac{3\xi_1}{|\xi_L-\xi_2|^5} - \frac{\xi_2}{|\xi_L-\xi_1|^3} + \frac{\xi_1}{|\xi_L-\xi_2|^3} - 1 \right\}$$

$$= K\left\{ \frac{2\xi_2}{|\xi_L-\xi_1|^3} - \frac{2\xi_1}{|\xi_L-\xi_2|^3} - 1 \right\}$$

Since $\xi_2 < 0$ and $\xi_1 > 0$ it follows that each term is always less than zero.

So,

$$\boxed{\frac{\partial^2 V'}{\partial \xi^2} < 0}$$

$$\frac{\partial^2 V'}{\partial \eta^2} = K\left\{ \frac{-\xi_2}{|\xi_L-\xi_1|^3} + \frac{\xi_1}{|\xi_L-\xi_2|^3} + 1 \right\}$$

Since $\xi_2 < 0$ and $\xi_1 > 0$ it follows that each term is always greater than zero

So,

$$\boxed{\frac{\partial^2 V'}{\partial \eta^2} > 0}$$

$$\frac{\partial^2 V'}{\partial \xi \partial \eta} = K\{0\} \quad \{\eta = 0 \text{ at points A, B, and C}\}$$

$$\boxed{\frac{\partial^2 V'}{\partial \xi \partial \eta} = 0}$$

Part 2.

Equations of motion for m:

$$\ddot{x} - 2\omega\dot{y} = -\frac{M_1 G(x-x_1)}{[(x-x_1)^2+y^2]^{3/2}} - \frac{M_2 G(x-x_2)}{[(x-x_2)^2+y^2]^{3/2}} + \frac{(M_1+M_2)Gx}{a^3}$$

$$\ddot{y} + 2\omega\dot{x} = -\frac{M_1 G y}{[(x-x_1)^2+y^2]^{3/2}} - \frac{M_2 G y}{[(x-x_2)^2+y^2]^{3/2}} + \frac{(M_1+M_2)Gy}{a^3}$$

Since a conservative force exists

$$\ddot{x} - 2\omega\dot{y} = -\frac{\partial 'V'}{\partial x}$$

$$\ddot{y} + 2\omega\dot{x} = -\frac{\partial 'V'}{\partial y}$$

where

$$'V' = -\frac{M_1 G}{[(x-x_1)^2+y^2]^{1/2}} - \frac{M_2 G}{[(x-x_2)^2+y^2]^{1/2}} - \frac{\omega^2}{2}(x^2+y^2)$$

It's noteworthy to show a constant of motion exists.

$$\ddot{x}\dot{x} - 2\omega\dot{y}\dot{x} = -\dot{x}\frac{\partial 'V'}{\partial x}$$

$$\ddot{y}\dot{y} + 2\omega\dot{y}\dot{x} = -\dot{y}\frac{\partial 'V'}{\partial y}$$

Summing these two equations gives

$$\ddot{x}\dot{x} + -2\omega\dot{y}\dot{x} + \ddot{y}\dot{y} + 2\omega\dot{y}\dot{x} = -\dot{x}\frac{\partial 'V'}{\partial x} - \dot{y}\frac{\partial 'V'}{\partial y}$$

$$\ddot{x}\dot{x} + \ddot{y}\dot{y} = -\frac{dx}{dt}\frac{\partial 'V'}{\partial x} - \frac{dy}{dt}\frac{\partial 'V'}{\partial y}$$

$$= -\frac{d'V'}{dt}$$

$$\frac{d}{dt}\left[\frac{1}{2}(\dot{x}^2+\dot{y}^2)\right] = -\frac{d'V'}{dt}$$

$$\frac{d}{dt}\left[\frac{1}{2}(\dot{x}^2+\dot{y}^2) + 'V'\right] = 0$$

$$\frac{d}{dt}'E' = 0 \quad \left\{ 'E' = \frac{1}{2}(\dot{x}^2+\dot{y}^2) + 'V' \right\}$$

Therefore, the total energy, 'E', in the moving coordinate system is a constant of the motion.

The Taylor expansion is about a point $(\xi_{A,B}, 0)$

$$'V'(\xi,\eta) \approx 'V'(\xi_{A,B},0) + 'V'_{\xi}(\xi_{A,B},0) + 'V'_{\eta}(\xi_{A,B},0) + \frac{1}{2}'V'_{\xi\xi}(\xi_{A,B},0)(\xi-\xi_{A,B})^2 + \frac{1}{2}'V'_{\xi\eta}(\xi_{A,B},0)(\xi-\xi_{A,B})(\eta) + \frac{1}{2}'V'_{\eta\eta}(\xi_{A,B},0)(\eta)^2$$

The equations of motion can be rewritten using the substitutions

$$\xi = \frac{x}{a}, \quad \eta = \frac{y}{a}, \quad \xi_1 = \frac{x_1}{a} = \frac{M_2}{M_1+M_2}, \quad \xi_2 = \frac{x_2}{a} = -\frac{M_1}{M_1+M_2}$$

$$\ddot{x} - 2w\dot{y} = -\frac{\partial 'V'}{\partial x} \longrightarrow \ddot{\xi}a - 2w\dot{\eta}a = -\frac{\partial 'V'}{\partial \xi}a$$

$$\ddot{y} + 2w\dot{x} = -\frac{\partial 'V'}{\partial y} \longrightarrow \ddot{\eta}a - 2w\dot{\xi}a = -\frac{\partial 'V'}{\partial \eta}a$$

Giving

$$\ddot{\xi} - 2w\dot{\eta} = -\frac{\partial 'V'}{\partial \xi}$$

$$\ddot{\eta} + 2w\dot{\xi} = -\frac{\partial 'V'}{\partial \eta}$$

Define $'V'_{\xi\xi} = \frac{\partial^2 'V'}{\partial \xi^2}(\xi_{A,B},0)$ and $'V'_{\eta\eta} = \frac{\partial^2 'V'}{\partial \eta^2}(\xi_{A,B},0)$.

Also,

$$'V'_{\xi}(\xi_{A,B},0) = 'V'_{\eta}(\xi_{A,B},0) = 'V'_{\xi\eta}(\xi_{A,B},0) = 0$$

So, the expanded $'V'(\xi,\eta)$ first derivative is

$$\frac{\partial 'V'}{\partial \xi} = 'V'_{\xi\xi}(\xi-\xi_{AB}) = 'V'_{\xi\xi}(\xi'), \quad \frac{\partial 'V'}{\partial \eta} = 'V'_{\eta\eta}(\eta)$$

Now, the equations of motion are

$$\ddot{\xi} - 2w\dot{\eta} = -'V'_{\xi\xi}(\xi') \longrightarrow \ddot{\xi} - 2w\dot{\eta} + 'V'_{\xi\xi}(\xi') = 0$$

$$\ddot{\eta} + 2w\dot{\xi} = -'V'_{\eta\eta}(\eta) \longrightarrow \ddot{\eta} + 2w\dot{\xi} + 'V'_{\eta\eta}(\eta) = 0$$

$$\xi' = \xi - \xi_{A,B}$$

$$\dot{\xi}' = \dot{\xi}, \quad \ddot{\xi}' = \ddot{\xi}$$

$$\ddot{\xi}' - 2\omega\dot{n} + {}'V'_{\xi\xi}\xi' = 0$$
$$\ddot{n} + 2\omega\dot{\xi} + {}'V'_{nn} n = 0$$

Let
$$\xi' = C_1 e^{pt}, \quad n = C_2 e^{pt}$$

$$C_1 p^2 - 2\omega C_2 p + C_1 {}'V'_{\xi\xi} = 0 \longrightarrow C_1(p^2 + {}'V'_{\xi\xi}) = 2\omega C_2 p$$
$$C_2 p^2 + 2\omega C_1 p + C_2 {}'V'_{nn} = 0 \longrightarrow C_2(p^2 + {}'V'_{nn}) = -2\omega C_1 p$$

$$\frac{C_2}{C_1} = \frac{p^2 + {}'V'_{\xi\xi}}{2\omega p}, \quad \frac{C_2}{C_1} = \frac{-2\omega p}{p^2 + {}'V'_{nn}}$$

$$\frac{p^2 + {}'V'_{\xi\xi}}{2\omega p} = \frac{-2\omega p}{p^2 + {}'V'_{nn}}$$

$$(p^2 + {}'V'_{\xi\xi})(p^2 + {}'V'_{nn}) = -4\omega^2 p^2$$
$$p^4 + p^2({}'V'_{\xi\xi} + {}'V'_{nn} + 4\omega^2) + {}'V'_{\xi\xi}{}'V'_{nn} = 0$$

Let $\lambda = p^2$

$$\lambda^2 + \lambda({}'V'_{\xi\xi} + {}'V'_{nn} + 4\omega^2) + {}'V'_{\xi\xi}{}'V'_{nn} = 0$$

Since the solution for p must be imaginary for stability then λ must have real negavite roots.

Let
$$A = \frac{\xi_1}{\rho_2^3} - \frac{\xi_2}{\rho_1^3} \qquad \rho_1^2 = (\xi - \xi_1)^2 + n^2, \quad \rho_2^2 = (\xi - \xi_2)^2 + n^2$$
$$B = 3\left[\frac{\xi_2(\xi - \xi_1)^2}{\rho_1^5} - \frac{\xi_1(\xi - \xi_2)^2}{\rho_2^5}\right]$$
$$C = 3\left[\frac{\xi_2}{\rho_1^5} - \frac{\xi_1}{\rho_2^5}\right]n^2$$

Then since

$${}^{1}V'_{\xi\xi} = 2K\left\{ \frac{\xi_2}{\rho_1^3} - \frac{\xi_1}{\rho_2^3} - 1 \right\}$$

it follows that

$${}^{1}V'_{\xi\xi} = -2A - 1$$

Also,

$${}^{1}V'_{nn} = K\left\{ \frac{-\xi_2}{\rho_1^3} + \frac{\xi_1}{\rho_2^3} + 1 \right\}$$

$${}^{1}V'_{nn} = A - 1$$

Now we have

$$\lambda^2 + \lambda(-2A - 1 + A - 1 + 4) + (1-A)(2A+1) = 0 \quad \{\omega = 1\}$$

$$\lambda^2 + \lambda(2-A) + (1-A)(2A+1) = 0$$

$$\lambda = \frac{A - 2 \pm \sqrt{(2-A)^2 - 4(1-A)(2A+1)}}{2}$$

$$= \frac{A - 2 \pm \sqrt{4 - 4A + A^2 - 4(2A + 1 - 2A^2 - A)}}{2}$$

$$= \frac{A - 2 \pm \sqrt{4 - 4A + A^2 - 8A - 4 + 8A^2 + 4A}}{2}$$

$$= \frac{A - 2 \pm \sqrt{9A^2 - 8A}}{2}$$

$$\lambda = \frac{A - 2 \pm \sqrt{A(9A-8)}}{2}$$

$$A - 2 + \sqrt{A(9A-8)} \le 0 \quad \text{and} \quad A - 2 - \sqrt{A(9A-8)} \le 0$$

$$A - 2 \le -\sqrt{A(9A-8)} \quad \text{and} \quad A - 2 \le \sqrt{A(9A-8)}$$

$$A^2 - 4A + 4 \le 9A^2 - 8A$$

$$-8A^2 + 4A + 4 \le 0$$

$$-4(2A^2 - A - 1) \le 0$$

$$2A^2 - A - 1 \le 0$$

$$(2A+1)(A-1) \le 0$$

$$A \le 1$$

∴ The discriminant must be greater than zero

$$9A^2 - 8A \geq 0$$

$$9A^2 \geq 8A$$

$$A \geq \frac{8}{9}$$

Therefore,

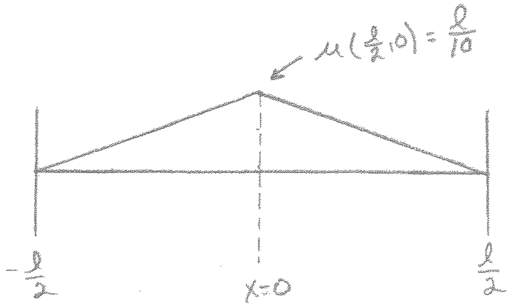
$$\frac{8}{9} \leq A \leq 1$$

$$\frac{8}{9} \leq \frac{\xi_1}{\rho_2^3} - \frac{\xi_2}{\rho_1^3} \leq 1$$

$$\frac{8}{9} \leq \frac{\xi_1}{|\xi_{A,B} - \xi_2|^3} - \frac{\xi_2}{|\xi_{A,B} - \xi_1|^3} \leq 1$$

$$\boxed{\frac{8}{9} \leq \frac{\xi_1}{|\xi_{A,B} - \xi_1 + 1|^3} - \frac{\xi_1 - 1}{|\xi_{A,B} - \xi_1|^3} \leq 1}$$

8-11



$$u(x,t) = f(x-ct) + g(x+ct)$$
$$= f(\xi) + g(\eta) \quad , \ \xi = x-ct \ , \ \eta = x+ct$$

$$u(x,0) = u_0(x) = \begin{cases} 0, & |x| > \frac{\ell}{2} \\ \frac{\ell}{10}\left(1 + \frac{2x}{\ell}\right), & -\frac{\ell}{2} \le x \le 0 \\ \frac{\ell}{10}\left(1 - \frac{2x}{\ell}\right), & 0 \le x \le \frac{\ell}{2} \end{cases}$$

1) $f(x) + g(x) = u_0(x)$

$$-c\frac{df}{d\xi} + c\frac{dg}{d\eta} = v_0(x) = 0$$

2) $f(x) = g(x) + C$

Substituting 2) into 1) gives

$$2g(x) + C = u_0(x)$$
$$g(x) = \frac{1}{2}u_0(x) - \frac{C}{2}$$

Solving for g(x) in 2) and substituting into 1) gives

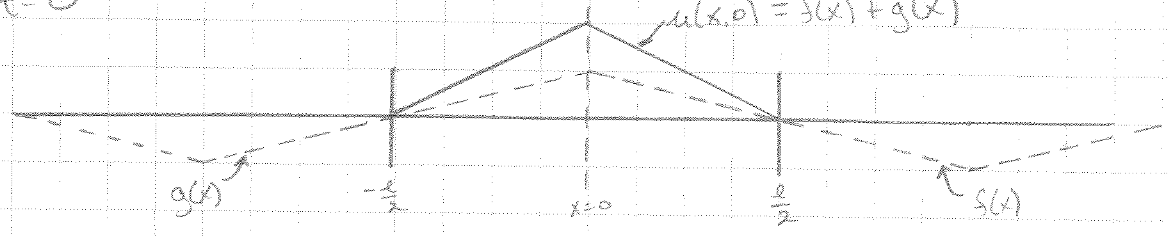
$$2f(x) - C = u_0(x)$$
$$f(x) = \frac{1}{2}u_0(x) + \frac{C}{2}$$

$$u(x,t) = f(x-ct) + g(x+ct)$$
$$u(x,t) = \frac{1}{2}u_0(x-ct) + \frac{1}{2}u_0(x+ct)$$

Thus, $u(x,t)$ represents two triangular waveforms, one traveling to the right and one traveling to the left.

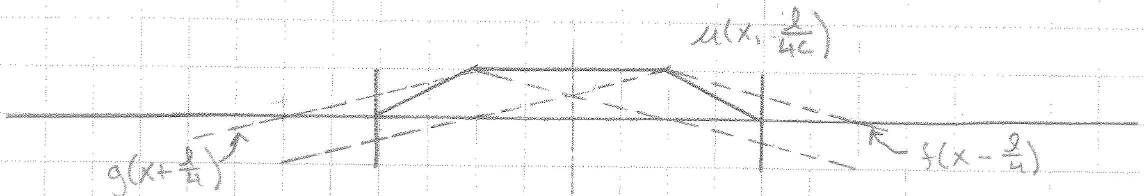
$$u(x,t) = \frac{\ell}{20}\left(1 - \frac{2(x-ct)}{\ell}\right)_{0 \le x \le \frac{\ell}{2}} + \frac{\ell}{20}\left(1 + \frac{2(x+ct)}{\ell}\right)_{-\frac{\ell}{2} \le x \le 0}$$

At $t=0$

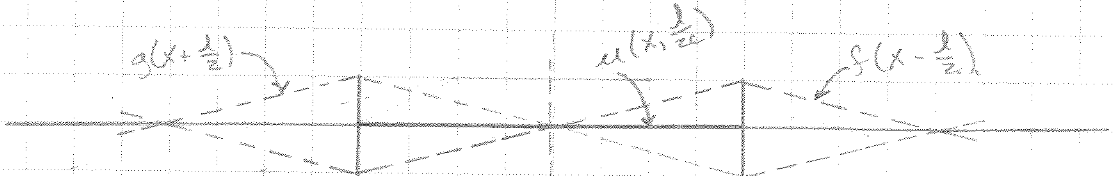


$$T = \frac{\lambda}{c} = \frac{2l}{c}$$

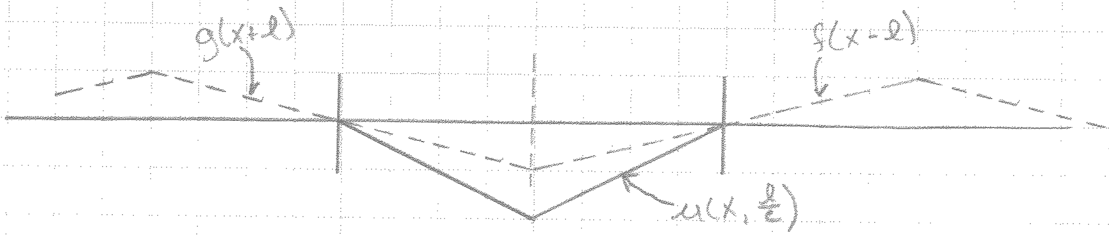
At $t = \frac{1}{4}\frac{l}{c}$ each waveform has shifted $\frac{l}{4}$ from $x=0$



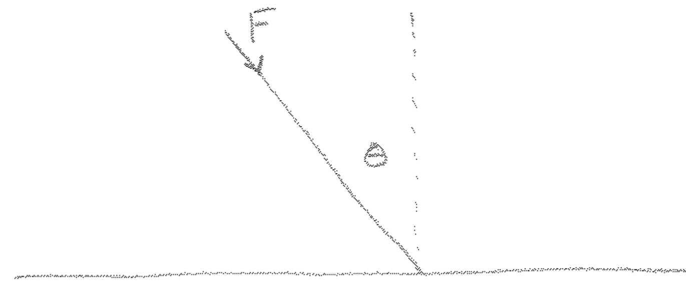
At $t = \frac{1}{2}\frac{l}{c}$ each waveform has shifted $\frac{l}{2}$ from $x=0$



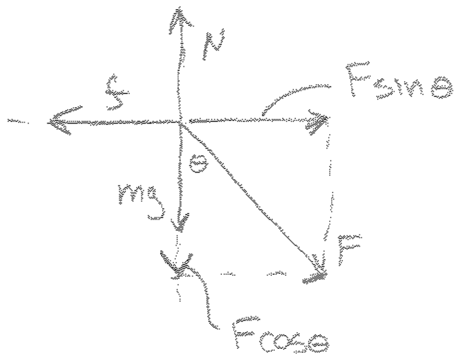
At $t = \frac{l}{c}$ each waveform has shifted l from $x=0$



1-7



a)



b) $V_x = 0 \rightarrow a_x = 0 \rightarrow \sum F_x = 0$

$0 = F\sin\theta - f$ $\qquad$ $f = \mu N \quad N = F\cos\theta + mg$

$\qquad\qquad\qquad\qquad\qquad = \mu F\cos\theta + \mu mg$

$F\sin\theta = f$

$\quad = \mu F\cos\theta + \mu mg$

$F(\sin\theta - \mu\cos\theta) = \mu mg$

$$\boxed{F = \frac{\mu mg}{\sin\theta - \mu\cos\theta}}$$

c) $\tan\theta_r = \mu_s \rightarrow \sin\theta_r = \mu_s\cos\theta_r \quad \cos\theta_r = \frac{\sin\theta_r}{\mu_s}$

Neglecting mass of mop

$f_s = \mu_s N \quad N = F\cos\theta_r$ $\qquad$ $F_x = F\sin\theta_r$

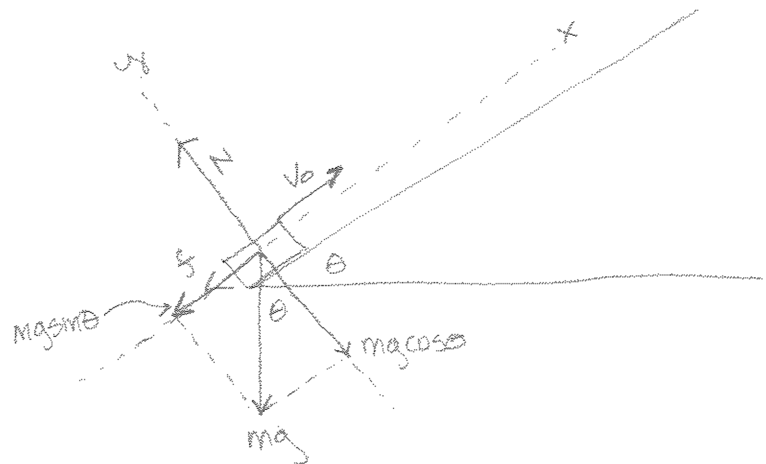
$= \mu_s F\cos\theta_r$ $\qquad\qquad\qquad = F\mu_s\cos\theta_r$

$\sum F_x = F_x - f$

$\quad = F\mu_s\cos\theta_r - \mu_s F\cos\theta_r$

$\sum F_x = 0$

1-9



$$ma = -mg\sin\theta - f \qquad f = \mu N$$
$$ma = -mg\sin\theta - \mu mg\cos\theta \qquad = \mu mg\cos\theta$$

Distance up ramp:

$$v^2 = v_0^2 + 2a(x - x_0)$$

$$0 = v_0^2 + 2a(x - x_0)$$

$$x - x_0 = \frac{v_0^2}{-2a}$$

$$x - x_0 = \frac{v_0^2}{2g(\sin\theta + \mu\cos\theta)}$$

Time up ramp:

$$x - x_0 = v_0 t + \frac{1}{2}at^2$$

$$0 = -(x - x_0) + v_0 t + \frac{1}{2}at^2$$

$$at^2 + 2v_0 t - 2(x - x_0) = 0$$

$$-g(\sin\theta + \mu\cos\theta)t^2 + 2v_0 t - \frac{v_0^2}{g(\sin\theta + \mu\cos\theta)} = 0$$

$$g(\sin\theta + \mu\cos\theta)t^2 - 2v_0 t + \frac{v_0^2}{g(\sin\theta+\mu\cos\theta)} = 0$$

$$g^2(\sin\theta + \mu\cos\theta)^2 t^2 - 2gv_0(\sin\theta+\mu\cos\theta) + v_0^2 = 0$$

$$t_u = \frac{2gv_0(\sin\theta+\mu\cos\theta) \pm \sqrt{4g^2v_0^2(\sin\theta+\mu\cos\theta)^2 - 4g^2v_0^2(\sin\theta+\mu\cos\theta)^2}}{2g^2(\sin\theta+\mu\cos\theta)^2}$$

$$t_u = \frac{v_0}{g}(\sin\theta+\mu\cos\theta)^{-1}$$

Time down the ramp

$$\Sigma F_x = -mg\sin\theta + f$$

$$ma = -mg\sin\theta + \mu mg\cos\theta$$

$$a = -g\sin\theta + \mu g\cos\theta$$

$$x_0 = \frac{v_0^2}{2g(\sin\theta+\mu\cos\theta)}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2$$

$$0 = x_0 + v_0 t + \frac{1}{2}at^2$$

$$at^2 + 2v_0 t + 2x_0 = 0 \qquad v_0 = 0$$

$$-g(\sin\theta - \mu\cos\theta)t^2 + \frac{v_0^2}{g(\sin\theta+\mu\cos\theta)} = 0$$

$$-g(\sin\theta - \mu\cos\theta)t^2 = \frac{-v_0^2}{g(\sin\theta+\mu\cos\theta)}$$

$$t^2 = \frac{v_0^2}{g^2(\sin\theta-\mu\cos\theta)(\sin\theta+\mu\cos\theta)}$$

$$t_d = \frac{v_0}{g}(\sin^2\theta - \mu^2\cos^2\theta)^{-1/2}$$

$$t = t_u + t_d$$

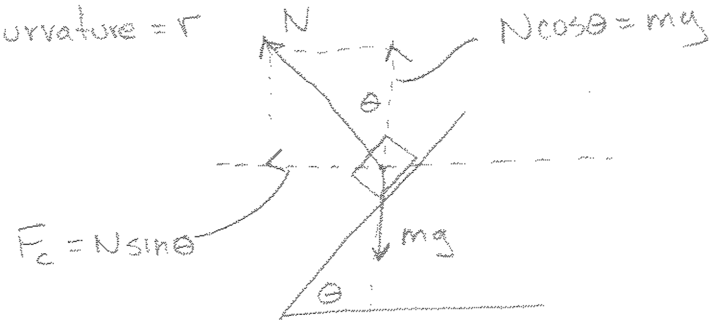
$$\boxed{t = \frac{v_0}{g}\left[(\sin\theta+\mu\cos\theta)^{-1} + (\sin^2\theta - \mu^2\cos^2\theta)^{-1/2}\right]}$$

1-10

Banked Curve Velocities

1) No Friction

Radius of curvature = r



$$F_c = N\sin\theta$$

$$\frac{mv^2}{r} = N\sin\theta \qquad N\cos\theta = mg$$

$$N = \frac{mg}{\cos\theta}$$

$$= mg\frac{\sin\theta}{\cos\theta}$$

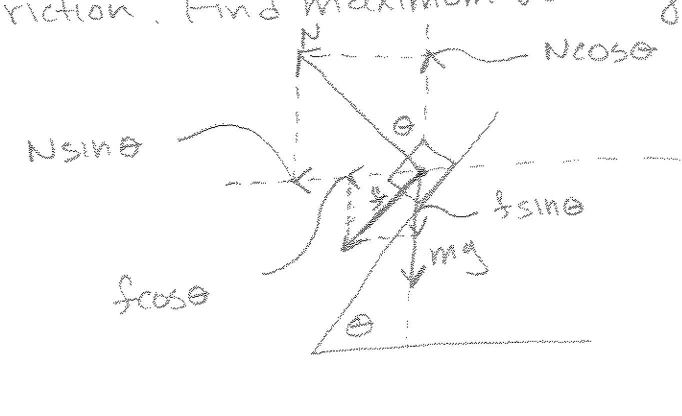
$$\frac{mv^2}{r} = mg\tan\theta$$

$$\frac{v^2}{r} = g\tan\theta$$

$$v^2 = gr\tan\theta$$

$$\boxed{v = (gr\tan\theta)^{1/2}} \text{ (ideal velocity)}$$

2) Friction: Find maximum velocity



$$F_c = N\sin\theta + f\cos\theta$$

$$= N\sin\theta + \mu N\cos\theta$$

$$= N(\sin\theta + \mu\cos\theta)$$

$$N\cos\theta = mg + f\sin\theta$$

$$= mg + \mu N\sin\theta$$

$$N\cos\theta - \mu N\sin\theta = mg$$

$$N(\cos\theta - \mu\sin\theta) = mg$$

$$N = \frac{mg}{\cos\theta - \mu\sin\theta}$$

$$= \frac{mg(\sin\theta + \mu\cos\theta)}{\cos\theta - \mu\sin\theta}$$

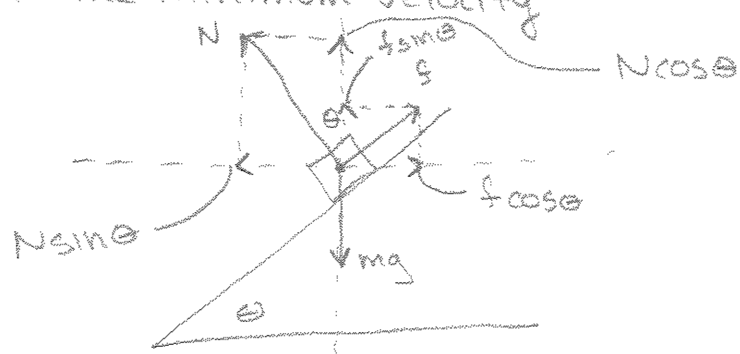
$$F_c = mg\,\frac{\tan\theta + \mu}{1 - \mu\tan\theta}$$

$$\frac{mv^2}{r} = mg\,\frac{\tan\theta + \mu}{1 - \mu\tan\theta}$$

$$v^2 = gr\,\frac{\tan\theta + \mu}{1 - \mu\tan\theta}$$

$$\boxed{v = \left(gr\,\frac{\tan\theta + \mu}{1 - \mu\tan\theta}\right)^{1/2}}$$

3) Friction: Find minimum velocity



$$F_c = N\sin\theta - f\cos\theta$$

$$= N\sin\theta - \mu N\cos\theta$$

$$= N(\sin\theta - \mu\cos\theta)$$

$$= mg\,\frac{(\sin\theta - \mu\cos\theta)}{(\cos\theta + \mu\sin\theta)}$$

$$F_c = mg\,\frac{(\tan\theta - \mu)}{(1 + \mu\tan\theta)}$$

$$\frac{mv^2}{r} = mg\,\frac{(\tan\theta - \mu)}{(1 + \mu\tan\theta)}$$

$$v^2 = gr\,\frac{\tan\theta - \mu}{1 + \mu\tan\theta}$$

$$\boxed{v = \left(gr\,\frac{\tan\theta - \mu}{1 + \mu\tan\theta}\right)^{1/2}}$$

$$mg = N\cos\theta + f\sin\theta$$

$$= N\cos\theta + \mu N\sin\theta$$

$$= N(\cos\theta + \mu N\sin\theta)$$

$$N = \frac{mg}{\cos\theta + \mu\sin\theta}$$

3-45 $\quad V(r) = \frac{1}{2}kr^2$

$$\int_{r_0'}^{r'} \frac{dr}{\left[E - V(r) - \frac{L^2}{2mr^2}\right]^{1/2}} = \sqrt{\frac{2}{m}}\,t$$

$$E - V(r) - \frac{L^2}{2mr^2} = E - \frac{1}{2}kr^2 - \frac{L^2}{2mr^2}$$

$$= \frac{2Emr^2 - kmr^4 - L^2}{2mr^2}$$

$$= \frac{-kmr^4 + 2Emr^2 - L^2}{2mr^2}$$

$$= \frac{-km\left(r^4 - 2\frac{E}{k}r^2 + \frac{L^2}{km}\right)}{2mr^2}$$

$$= \frac{-k}{2r^2}\left(r^4 - \frac{2E}{k}r^2 + \frac{E^2}{k^2} + \frac{L^2}{km} - \frac{E^2}{k^2}\right)$$

$$= \frac{-k}{2r^2}\left[\left(r^2 - \frac{E}{k}\right)^2 - \frac{E^2}{k^2} + \frac{L^2}{km}\right]$$

$$= \frac{k}{2r^2}\left[\frac{E^2}{k^2} - \frac{L^2}{km} - \left(r^2 - \frac{E}{k}\right)^2\right]$$

$$\int_{r_0'}^{r'} \frac{dr}{\left[E - V(r) - \frac{L^2}{2mr^2}\right]^{1/2}} = \int_{r_0'}^{r'} \frac{dr}{\left[\frac{k}{2r^2}\left(\frac{E^2}{k^2} - \frac{L^2}{km} - \left[r^2 - \frac{E}{k}\right]^2\right)\right]^{1/2}}$$

$$= \sqrt{\frac{2}{k}} \int_{r_0'}^{r'} \frac{r\,dr}{\left[\frac{E^2}{k^2} - \frac{L^2}{km} - \left(r^2 - \frac{E}{k}\right)^2\right]^{1/2}}$$

let $r^2 - \frac{E}{k} = \frac{\sqrt{E^2 - \frac{L^2k}{m}}}{k}\cos 2\alpha$

$$2r\,dr = -\frac{\sqrt{E^2 - \frac{L^2k}{m}}}{k}\sin 2\alpha \cdot 2\,d\alpha$$

$$-r\,dr = \frac{\sqrt{E^2 - \frac{L^2k}{m}}}{k}\sin 2\alpha\,d\alpha$$

$$= -\sqrt{\frac{2}{k}} \int_{\alpha_0}^{\alpha} \frac{\frac{\sqrt{E^2 - \frac{L^2k}{m}}}{k}\sin 2\alpha'\,d\alpha'}{\frac{\sqrt{E^2 - \frac{L^2k}{m}}}{k}\left(1 - \cos^2 2\alpha'\right)^{1/2}}$$

$$= -\sqrt{\frac{2}{k}} \int_{\alpha_0}^{\alpha} d\alpha'$$

$$= -\sqrt{\frac{2}{k}}(\alpha - \alpha_0)$$

$$-\sqrt{\frac{2}{k}}(\alpha - \alpha_0) = \sqrt{\frac{2}{m}}\,t$$

$$-(\alpha - \alpha_0) = \sqrt{\frac{k}{m}}\,t$$

$$= \omega t$$

$$-\alpha = \omega t + \alpha_0$$

$\cos(-\alpha) = \cos(\alpha)$ {even function}

$$r^2 - \frac{E}{k} = \frac{\left(E^2 - \frac{L^2k}{m}\right)^{1/2}}{k}\cos 2\alpha$$

$$r^2 = \frac{E}{k} + \frac{\left(E^2 - L^2\omega^2\right)^{1/2}}{k}\cos 2(\omega t + \alpha_0)$$

$$\boxed{kr^2 = E + \left(E^2 - L^2\omega^2\right)^{1/2}\cos(2\omega t + 2\alpha_0), \quad \omega = \sqrt{\frac{k}{m}}}$$

$$\Theta = \Theta_0 + \int_0^t \frac{L}{mr^2}\,dt$$

$$\Theta - \Theta_0 = \int_0^t \frac{L}{\frac{m}{K}\left\{E + (E^2 - L^2\omega^2)^{1/2}\cos(2\omega t + 2\alpha_0)\right\}}\,dt \quad , \quad \frac{m}{K} = \frac{1}{\omega^2}$$

$$= \int_0^t \frac{L\omega^2}{E + (E^2 - L^2\omega^2)^{1/2}\cos(2\omega t + 2\alpha_0)}\,dt$$

Let $a = E$, $b = (E^2 - L^2\omega^2)^{1/2}$

$$\Theta - \Theta = L\omega^2 \int_0^t \frac{dt}{a + b\cos(2(\omega t + \alpha_0))}$$

let $u = \tan(\frac{X}{2})$, $X = 2(\omega t + \alpha_0)$

$du = \frac{1}{2}\sec^2(\frac{X}{2})\,2\omega\,dt$

$= (1 + \tan^2(\frac{X}{2}))\,\omega\,dt$

$= (1 + u^2)\,\omega\,dt$

$\frac{du}{\omega(1+u^2)} = dt$

$$= \frac{L\omega^2}{\omega}\int_{u_0}^{u} \frac{du}{(1+u^2)\left(a + b\frac{(1-u^2)}{1+u^2}\right)}$$

$$= L\omega \int_{u_0}^{u} \frac{du}{a(1+u^2) + b(1-u^2)}$$

$$= L\omega \int_{u_0}^{u} \frac{du}{a + au^2 + b - bu^2}$$

$\cos X = \frac{1-u^2}{1+u^2}$ {this is THE TRICK}

$$= L\omega \int_{u_0}^{u} \frac{du}{(a+b) + (a-b)u^2}$$

$$= \frac{L\omega}{a-b}\int_{u_0}^{u} \frac{du}{\frac{a+b}{a-b} + u^2}$$

let $c^2 = \frac{(a+b)}{(a-b)}$

$$= \frac{L\omega}{a-b}\int_{u_0}^{u} \frac{du}{c^2 + u^2}$$

let $u = c\tan\beta$

$du = c\sec^2\beta\,d\beta$

$$= \frac{L\omega}{a-b}\int_{\beta_0}^{\beta} \frac{c\sec^2\beta\,d\beta}{c^2(1+\tan^2\beta)}$$

$$= \frac{L\omega}{(a-b)c}\int_{\beta_0}^{\beta} d\beta$$

$$= \frac{L\omega}{(a-b)c}\tan^{-1}\left(\frac{u}{c}\right)\Big|_{u_0}^{u}$$

$$= \frac{L\omega}{(a-b)c}\tan^{-1}\left(\frac{\tan\frac{X}{2}}{\left(\frac{a+b}{a-b}\right)^{1/2}}\right)\Bigg|_{X_0}^{X}$$

$$\theta - \theta_0 = \frac{L\omega}{(a-b)c} \tan^{-1}\left(\frac{\tan(\omega t + \alpha_0)}{\left(\frac{a+b}{a-b}\right)^{1/2}} \right)\Bigg|_0^t$$

$$(a-b)c = [(a-b)(a+b)]^{1/2}$$
$$= [a^2 - b^2]^{1/2}$$
$$= [E^2 - (E^2 - L^2\omega^2)]^{1/2}$$
$$= L\omega$$

$$\theta - \theta_0 = \frac{L\omega}{L\omega} \tan^{-1}\left(\frac{\tan(\omega t + \alpha_0)}{\left(\frac{a+b}{a-b}\right)^{1/2}} \right)\Bigg|_0^t$$

$$\theta - \theta_0 = \tan^{-1}\left(\tan(\omega t + \alpha_0) \left(\frac{a-b}{a+b}\right)^{1/2} \right)\Bigg|_0^t$$

$$\left(\frac{a-b}{a+b}\right)^{1/2} = \left[\frac{a-b}{a+b} \cdot \frac{a-b}{a-b}\right]^{1/2}$$
$$= \left[\frac{(a-b)^2}{a^2-b^2}\right]^{1/2}$$
$$= \frac{a-b}{(a^2-b^2)^{1/2}}$$
$$= \frac{E - (E^2 - L^2\omega^2)^{1/2}}{[E^2 - (E^2 - L^2\omega^2)]^{1/2}}$$
$$= \frac{E - (E^2 - L^2\omega^2)^{1/2}}{(L^2\omega^2)^{1/2}}$$
$$= \frac{E - (E^2 - L^2\omega^2)^{1/2}}{L\omega}$$

$$\theta - \theta_0 = \tan^{-1}\left(\tan(\omega t + \alpha_0) \left(\frac{E - (E^2 - L^2\omega^2)^{1/2}}{L\omega} \right) \right)$$

$$\boxed{\tan(\theta - \theta_0) = (L\omega)^{-1}\left[E - (E^2 - L^2\omega^2)^{1/2}\right] \tan(\omega t + \alpha_0)}$$

$$r^2 = x^2 + y^2 + z^2$$

$$kr^2 = E + (E^2 - L^2\omega^2)^{1/2} \cos(2\omega t + 2\alpha_0)$$

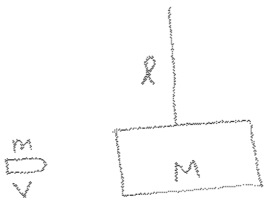
$$r^2 \propto \cos(2\omega t + 2\alpha_0)$$

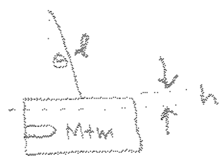
$$\tan(\theta - \theta_0) \propto \tan(\omega t + \alpha_0)$$

$$\theta - \theta_0 \propto \omega t + \alpha_0$$

$\omega_x = \omega_y = \omega_z$ {given as isotropic from Problem 3-43}
and $\omega_r = 2\omega_\theta$, the motion is an ellipse

4-3





$\cos\theta = \frac{x}{\ell}$

$x = \ell\cos\theta$

$h = \ell - x$

$= \ell - \ell\cos\theta$

$= \ell(1-\cos\theta)$

$E = T + V$

$E_i = \frac{1}{2}mv^2$

$E_f = (M+m)gh$

$= (M+m)g\ell(1-\cos\theta)$

$E_i = E_f$

$\frac{1}{2}mv^2 = (M+m)g\ell(1-\cos\theta)$

$v^2 = (1-\cos\theta)2g\ell\left(\frac{M+m}{m}\right)$

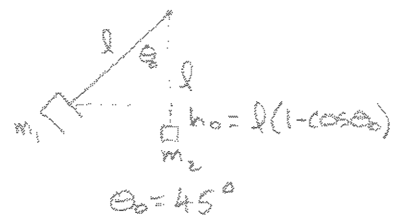
$v = \left[(1-\cos\theta)2g\ell\left(\frac{M+m}{m}\right)\right]^{1/2}$

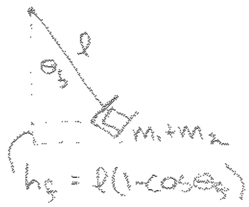
$2\sin^2\frac{\theta}{2} = (1-\cos\theta)$

$= \left[4\sin^2\frac{\theta}{2}\,g\ell\left(1+\frac{M}{m}\right)\right]^{1/2}$

$$\boxed{v = 2\left[\left(1+\frac{M}{m}\right)g\ell\right]^{1/2}\sin\frac{\theta}{2}}$$

4-5





$T + V = E$

$E_0 = mgh_0$

$= m_1 g l(1-\cos\theta_0)$

$E_f = (m_1+m_2)gh_f$

$= (m_1+m_2)gl(1-\cos\theta_f)$

Velocity just prior to impact

$\frac{1}{2} m_1 V_0^2 = E_0 = m_1 g h_0$

$V_0^2 = 2gh_0$

Angular momentum just prior to impact

$L_0 = m_1 V_0 l$

Angular momentum just after impact

$L_f = (m_1+m_2) V_f l$

Conservation of angular momentum

$L_0 = L_f$

$m_1 V_0 l = (m_1+m_2) V_f l$

$$V_f = V_0 \frac{m_1}{m_1+m_2}$$

Energy just after impact

$E_f = \frac{1}{2}(m_1+m_2) V_f^2$

Conservation of Energy after impact

$\frac{1}{2}(m_1+m_2)V_f^2 = (m_1+m_2) g l (1-\cos\theta_f)$

$1-\cos\theta_f = \frac{1}{2}\frac{V_f^2}{gl}$, $V_f^2 = V_0^2 \left(\frac{m_1}{m_1+m_2}\right)^2$

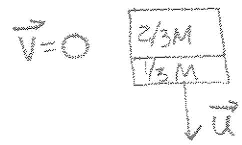
$= 2gl(1-\cos\theta_0)\left(\frac{m_1}{m_1+m_2}\right)^2$

$\cos\theta_f = 1 - (1-\cos\theta_0)\left(\frac{m_1}{m_1+m_2}\right)^2$

$\theta_f = \cos^{-1}\left[1 - (1-\cos\theta_0)\left(\frac{m_1}{m_1+m_2}\right)^2\right]$, $1-\cos\theta_0 = 1 - .707$
$= .293$

$$\boxed{\theta_f = \cos^{-1}\left[1 - .293\left(\frac{m_1}{m_1+m_2}\right)^2\right]}$$

4-7



$$F = \frac{M(t)g}{6}$$

$$\frac{d}{dt}(M\vec{v}) - \frac{dM}{dt}(\vec{v}+\vec{u}) = F$$

$$\vec{F} = -\frac{dM}{dt}\vec{u}$$

$$\frac{Mg}{6} = -\frac{dM}{dt}u$$

$$\frac{-g}{6u}dt = \frac{dM}{M}$$

$$\frac{-g}{6u}\int_0^t dt = \int_M^{2/3M}\frac{dM}{M}$$

$$\frac{-gt}{6u} = \ln M\Big|_M^{2/3M}$$

$$= \ln\frac{2M}{3} - \ln M$$

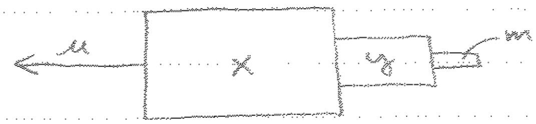
$$= \ln\frac{2}{3}$$

$$t = \frac{6u}{g}\ln\frac{3}{2} \quad , u = 1500\,m/s$$

$$= \frac{6(1500)}{9.81}\ln\frac{3}{2}$$

$$\boxed{t = 372\,s}$$

4-9



$V_0 = 0$
$V_5 = 6000$ m/s
$u = 1500$ m/s
$r = .10$

$$\frac{d}{dt}(M\vec{V}) - \frac{dM}{dt}(\vec{V}+\vec{u}) = F$$

$$\frac{dM}{dt}\vec{V} + M\frac{d\vec{V}}{dt} - \frac{dM}{dt}\vec{V} - \frac{dM}{dt}\vec{u} = 0 \quad \{F=0\}$$

$$M\frac{d\vec{V}}{dt} - \frac{dM}{dt}\vec{u} = 0$$

$$M\,dv = -u\,dM$$
$$dv = -u\frac{dM}{M}$$

$$\int_{V_0}^{V_1} dV = -u\int_{M_0}^{M_1}$$

$$V_1 - V_0 = u \ln \frac{M_0}{M_1}$$

a) First stage

$$V_1 = u \ln \frac{x+y+m}{rx+y+m}$$

Second stage

$$V_2 = u \ln \frac{y+m}{ry+m}$$

$$V_5 = V_1 + V_2$$

$$= u \ln\left[\frac{x+y+m}{rx+y+m}\cdot\frac{y+m}{ry+m}\right]$$

$$e^{\frac{V_5}{u}} = \frac{x+y+m}{rx+y+m}\cdot\frac{y+m}{ry+m}$$

$$(rx+y+m)(ry+m)e^{k} = (x+y+m)(y+m) \quad , K = \frac{y}{x}$$

$$rx(ry+m)e^{k} + (y+m)(ry+m)e^{k} = x(y+m) + (y+m)^{2}$$

$$(y+m)(ry+m)e^{k} - (y+m)^{2} = x(y+m) - rx(ry+m)e^{k}$$

$$= x\left(y+m - r(ry+m)e^{k}\right)$$

$$X = \frac{(y+m)(ry+m)e^{k} - (y+m)^{2}}{y+m-r(ry+m)e^{k}}$$

$$= \frac{(y+m)\left[(ry+m)e^{k} - (y+m)\right]}{y+m-r(ry+m)e^{k}}$$

$$M_{T} = x+y+m$$

$$= \frac{(y+m)\left[(ry+m)e^{k} - y - m\right]}{y+m-r(ry+m)e^{k}} + y + m$$

$$\frac{dM_{T}}{dy} = \frac{d}{dy}\left[\frac{(y+m)\left[(ry+m)e^{k} - (y+m)\right]}{y+m-r(ry+m)e^{k}}\right] + 1$$

$$\frac{dx}{dy} = \Big[\left[(y+m) - (ry+m)re^{k}\right]\cdot\left[(y+m)(re^{k}-1) + \left[(ry+m)e^{k} - (y+m)\right]\right] -$$
$$\left[(y+m)\left[(ry+m)e^{k} - (y+m)\right]\left[1-r^{2}e^{k}\right]\right]\Big]/D^{2}$$

$$D^{2}\frac{dx}{dy} = (y+m)^{2}(re^{k}-1) + (y+m)(ry+m)e^{k} - (y+m)^{2} - (y+m)(ry+m)(re^{k}-1)re^{k}$$
$$-(ry+m)^{2}re^{2k} + (y+m)(ry+m)re^{k} - \Big[(y+m)(ry+m)e^{k} -$$
$$(y+m)^{2} - (y+m)(ry+m)r^{2}e^{2k} + (y+m)^{2}r^{2}e^{k}\Big]$$

$$= (y+m)^{2}re^{k} - (y+m)^{2} + (y+m)(ry+m)e^{k} - (y+m)^{2} - (y+m)(ry+m)r^{2}e^{2k} +$$
$$(y+m)(ry+m)re^{k} - (ry+m)^{2}re^{2k} + (y+m)(ry+m)re^{k} -$$
$$(y+m)(ry+m)e^{k} + (y+m)^{2} + (y+m)(ry+m)r^{2}e^{2k} - (y+m)^{2}r^{2}e^{k}$$

$$= (y+m)^2 re^k - (y+m)^2 + 2(y+m)(ry+m)re^k - (ry+m)^2 re^{2k} - (y+m)^2 r^2 e^k$$

Minimize M_T

$$M_T = x + y + m$$

$$\frac{dM_T}{dy} = \frac{dx}{dy} + 1$$

$$0 = D^2 \frac{dx}{dy} + D^2$$

$$0 = (y+m)^2 re^k - (y+m)^2 + 2(y+m)(ry+m)re^k - (ry+m)^2 re^{2k} - (y+m)^2 r^2 e^k + ((y+m) - (ry+m)re^k)^2$$

$$= (y+m)^2 re^k - (y+m)^2 + 2(y+m)(ry+m)re^k - (ry+m)^2 re^{2k} - (y+m)^2 r^2 e^k + (y+m)^2 - 2(y+m)(ry+m)re^k + (ry+m)^2 r^2 e^{2k}$$

$$= (y+m)^2 re^k - (ry+m)^2 re^{2k} - (y+m)^2 r^2 e^k + (ry+m)^2 r^2 e^{2k}$$

$$= (y^2 + 2my + m^2) re^k - (r^2y^2 + 2rmy + m^2) re^{2k} - (y^2 + 2my + m^2) r^2 e^k + (r^2y^2 + 2rmy + m^2) r^2 e^{2k}$$

$$= y^2(re^k - r^3e^{2k} + r^2e^k + r^4e^{2k}) + y(2mre^k - 2mr^2e^{2k} - 2mr^2e^k + 2mr^3e^{2k}) + m^2re^k - m^2re^{2k} - m^2r^2e^k + m^2r^2e^{2k}$$

$$= y^2 re^k(1 + r - r^2e^k + r^3e^k) + y2mre^k(1 - re^k - r + r^2e^k) + m^2re^k(1 - e^k - r + re^k)$$

$$0 = y^2 + \frac{2mre^k(1 - re^k - r + r^2e^k)}{re^k(1 + r - r^2e^k + r^3e^k)} + \frac{m^2re^k(1 - e^k - r + re^k)}{re^k(1 + r - r^2e^k + r^3e^k)}$$

$$\left(y + \frac{m(1-re^k-r+r^2e^k)}{(1+r-r^2e^k+r^3e^k)}\right)^2 = \left[\frac{m(1-r-re^k+r^2e^k)}{(1+r-r^2e^k+r^3e^k)}\right]^2 -$$

$$\frac{m^2(1-r-e^k+re^k)}{(1+r-r^2e^k+r^3e^k)}$$

$$y = \frac{-m(1-r-re^k+r^2e^k)}{(1+r-r^2e^k+r^3e^k} \pm \left[\left[\frac{m(1-r-re^k+r^2e^k)}{(1+r-r^2e^k+r^3e^k)}\right]^2 + \frac{m^2(1+r+e^k-re^k)}{1+r-r^2e^k+r^3e^k}\right]^{1/2}$$

$$= m\left[\frac{-(1-r-re^k+r^2e^k)}{1+r-r^2e^k+r^3e^k} \pm \left[\left(\frac{1-r-re^k+r^2e^k}{1+r-r^2e^k+r^3e^k}\right)^2 + \frac{1+r+e^k-re^k}{1+r-r^2e^k+r^3e^k}\right]^{1/2}\right]$$

$$\frac{1-r-re^k+r^2e^k}{1+r-r^2e^k+r^3e^k} = \frac{-4.01}{.409} = -9.804$$

$$\frac{1+r+e^k-re^k}{1+r-r^2e^k+r^3e^k} = \frac{50.238}{.409} = 122.83$$

$$y = m\left[9.804 \pm \left[96.126 + 122.83\right]^{1/2}\right]$$

$$= m\left[9.804 + 14.80\right]$$

$$= m\left[24.60\right]$$

$$\boxed{y = 2460\ \text{kg}}$$

$$x = \frac{(y+m)(ry+m)e^k - (y+m)^2}{y+m - r(ry+m)e^k}$$

$$= \frac{(2460+100)(246+100)e^4 - (2460+100)^2}{2460+100 - .1(246+100)e^4}$$

$$= \frac{(2560)(346)e^4 - (2560)^2}{2560 - .1(346)e^4}$$

$$\boxed{x = 62315 \text{ kg}}$$

b) For a single stage

$$V_s = u \ln \frac{x+m}{rx+m}$$

$$e^{\frac{V_s}{u}} = \frac{x+m}{rx+m} \qquad K = \frac{V_s}{u} = 4$$

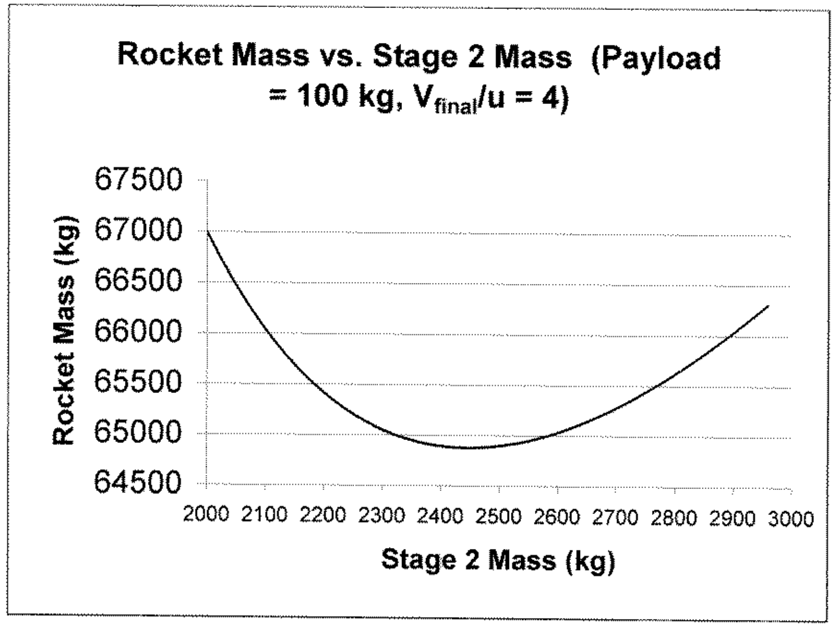
$$(rx+m)e^k = x+m$$
$$me^k - m = x - rxe^k$$
$$m(e^k - 1) = x(1 - re^k)$$
$$x = \frac{m(e^k-1)}{1-re^k}$$

$re^k < 1$ condition must hold. But $r = .1$ and $e^k = 54.6$. So

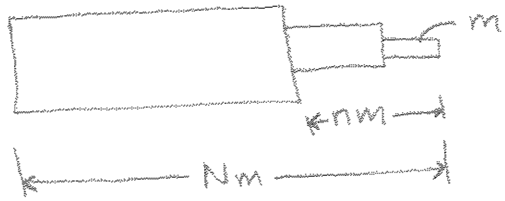
$$re^k = 5.46 > 1$$

Therefore, a single stage rocket cannot achieve 4 times the exhaust velocity when $r = 0.1$.

2442	41309237	662.7317	62331.77	64873.77
2444	41364430	663.6397	62329.65	64873.65
2446	41419658	664.5477	62327.59	64873.59
2448	41474923	665.4558	62325.59	64873.59
2450	41530222	666.3638	62323.65	64873.65
2452	41585558	667.2719	62321.76	64873.76
2454	41640929	668.1799	62319.94	64873.94
2456	41696336	669.0879	62318.17	64874.17
2458	41751779	669.996	62316.46	64874.46
2460	41807257	670.904	62314.81	64874.81
2462	41862771	671.812	62313.22	64875.22
2464	41918321	672.7201	62311.68	64875.68
2466	41973906	673.6281	62310.21	64876.21
2468	42029527	674.5362	62308.78	64876.78
2470	42085184	675.4442	62307.42	64877.42
2472	42140877	676.3522	62306.11	64878.11
2474	42196605	677.2603	62304.86	64878.86
2476	42252368	678.1683	62303.66	64879.66
2478	42308168	679.0763	62302.52	64880.52
2480	42364003	679.9844	62301.44	64881.44
2482	42419874	680.8924	62300.41	64882.41
2484	42475780	681.8005	62299.43	64883.43
2486	42531722	682.7085	62298.51	64884.51
2488	42587700	683.6165	62297.65	64885.65
2490	42643714	684.5246	62296.83	64886.83
2492	42699763	685.4326	62296.08	64888.08
2494	42755848	686.3406	62295.38	64889.38
2496	42811968	687.2487	62294.73	64890.73
2498	42868125	688.1567	62294.13	64892.13
2500	42924317	689.0647	62293.59	64893.59
2502	42980544	689.9728	62293.1	64895.1
2504	43036807	690.8808	62292.66	64896.66
2506	43093106	691.7889	62292.28	64898.28
2508	43149441	692.6969	62291.95	64899.95
2510	43205811	693.6049	62291.67	64901.67
2512	43262217	694.513	62291.45	64903.45
2514	43318659	695.421	62291.27	64905.27



4-9



$m = 100\text{kg}$

$V_0 = 0$

$V_f = 6000 \text{ m/s}$

$u = 1500 \text{ m/s}$

$\vec{F} = 0$

$$r = \frac{\text{empty canister}}{\text{empty canister + fuel}} = .10$$

$$\frac{d}{dt}(M\vec{V}) - \frac{dM}{dt}(\vec{V}+\vec{u}) = F$$

$$\frac{dM}{dt}\vec{V} + M\frac{d\vec{V}}{dt} - \frac{dM}{dt}\vec{V} - \frac{dM}{dt}\vec{u} = 0$$

$$M\frac{d\vec{V}}{dt} - \frac{dM}{dt}\vec{u} = 0$$

$$M\frac{dV}{dt} = -u\frac{dM}{dt}$$

$$dV = -u\frac{dM}{M}$$

$$\int_{V_0}^{V_1} dV = -u\int_{m_i}^{m_1}\frac{dM}{M}$$

$$V_1 = -u \ln\frac{m_1}{m_i}$$

$$V_1 = u\ln\frac{m_i}{m_1}$$

$m_i = Nm$

$m_1 = r(Nm - nm) + nm$

$= m[r(N-n)+n]$

$$V_1 = u\ln\frac{Nm}{m[r(N-n)+n]}$$

$$V_1 = u\ln\frac{N}{rN+n(1-r)}$$

$$V_2 = u\ln\frac{m_i'}{m_2}$$

$m_i' = nm$

$m_2 = r(nm - m) + m$

$= m[r(n-1)+1]$

$= m[rn - r + 1]$

$$= u\ln\frac{nm}{m[rn-r+1]}$$

$$V_2 = u\ln\frac{n}{rn-r+1}$$

$$V_s = V_1 + V_2$$

$$= u \ln\left(\frac{N}{rN + n(1-r)} \cdot \frac{n}{rn - r + 1}\right)$$

N and r are constants. To find n giving a maximum velocity, take $\frac{d}{dn}$ and set $\frac{dV_s}{dn} = 0$.

$$\frac{d}{dx} \ln u = \frac{1}{u}\frac{du}{dx}$$

$$\frac{d}{dn} \ln(f(n)) = \frac{1}{f(n)} \frac{d f(n)}{dn}$$

$$= \frac{(rN + n(1-r))(rn - r + 1)}{Nn} \cdot \left[\frac{(rN+n(1-r))(rn-r+1)N - Nn\frac{d}{dn}\left[(rN+n(1-r))(rn-r+1)\right]}{\left[(rN+n(1-r))(rn-r+1)\right]^2}\right]$$

$$= \frac{1}{Nn}\left[\frac{N}{D} - \frac{Nn\left[(rN+n(1-r))(r) + (rn-r+1)(1-r)\right]}{D^2}\right]$$

$$0 = \frac{1}{nD} - \frac{(rN+n(1-r))(r) + (rn-r+1(1-r)}{D^2}$$

$$\frac{(rN+n(1-r))(r) + (rn-r+1)(1-r)}{D^2} = \frac{1}{nD}$$

$$n\left[r^2N + rn(1-r) + (rn-r+1)(1-r)\right] = D$$

$$n\left[r^2N + rn - r^2n + rn - r + 1 - r^2n + r^2 - r\right] = D$$

$$n\left[r^2N + 2rn - 2r^2n - 2r + r^2 + 1\right] = D$$

$$r^2Nn + 2rn^2 - 2r^2n^2 - 2rn + r^2n + n = (rN + n(1-r))(rn - r + 1)$$

$$= (rN + n - nr)(rn - r + 1)$$

$$r^2\cancel{Nn} + 2rn^2 - 2r^2n^2 - \cancel{2rn} + \cancel{r^2n} + \cancel{n} = r^2\cancel{Nn} - r^2N + rN + rn^2 - \cancel{rn} + \cancel{n} - n^2r^2 + \cancel{rn} - \cancel{nr}$$

$$rn^2 - r^2n^2 = rN - r^2N$$

$$n^2(r - r^2) = N(r - r^2)$$

$$n^2 = N$$

$$\boxed{n = N^{1/2}}$$

$$v_f = u \ln\left(\frac{N}{rN + \sqrt{N}(1-r)} \cdot \frac{N^{1/2}}{rN^{1/2} - r + 1}\right)$$

$$= u \ln\left(\frac{N}{N^{1/2}(rN^{1/2}+1-r)} \cdot \frac{N^{1/2}}{rN^{1/2}+1-r}\right)$$

$$= u \ln\left(\frac{N^{1/2}}{rN^{1/2}+1-r} \cdot \frac{N^{1/2}}{rN^{1/2}+1-r}\right)$$

$$v_f = 2u \ln\left(\frac{N^{1/2}}{rN^{1/2}+1-r}\right)$$

$$\frac{v_f}{2u} = \ln\left(\frac{N^{1/2}}{rN^{1/2}+1-r}\right)$$

$$e^K = \frac{N^{1/2}}{rN^{1/2}+1-r} \quad , K = \frac{v_f}{2u}$$

$$e^k r N^{1/2} + e^k - re^k = N^{1/2}$$

$$e^k - re^k = N^{1/2} - e^k r N^{1/2}$$

$$= N^{1/2}(1 - re^k)$$

$$\frac{e^k - re^k}{1 - re^k} = N^{1/2}$$

$$\frac{e^k(1-r)}{1-re^k} = N^{1/2}$$

$$N = \frac{e^{2k}(1-r)^2}{(1-re^k)^2}$$

$$= \frac{e^4 (.9)^2}{(1-.1e^2)^2}$$

$2k = \frac{v_f}{u} = 4$

$k = 2$ {each stage provides half the final velocity}

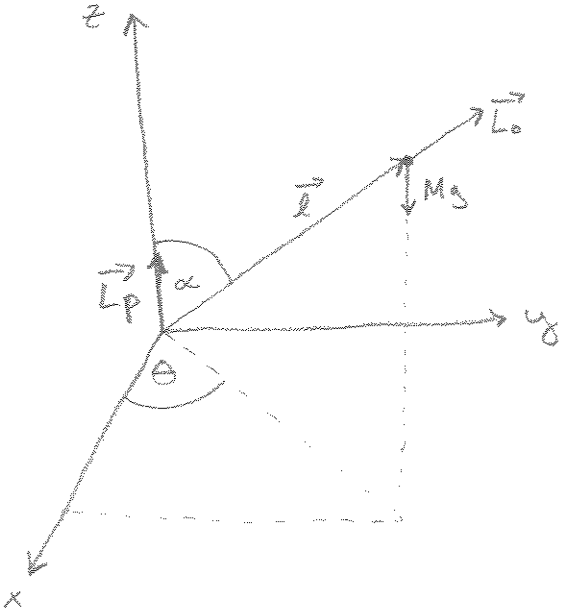
$$= \frac{44.22}{.068}$$

$N = 648.66 \longrightarrow$ $\boxed{1^{st} \text{ stage mass} = 64866\,\text{Kg} - 2547\,\text{Kg} = 62319\,\text{Kg}}$

$n = N^{1/2}$

$n = 25.47 \longrightarrow$ $\boxed{2^{nd} \text{ stage mass} = 2547\,\text{Kg} - 100\,\text{Kg} = 2447\,\text{Kg}}$

4-11



$$\vec{L} = \vec{L}_o + \vec{L}_p$$

$$\vec{L}_o = MR^2\omega_o(\cos\theta\sin\alpha\,\hat{i} + \sin\theta\sin\alpha\,\hat{j} + \cos\alpha\,\hat{k})$$

$$\vec{L}_p = M(l\sin\alpha)^2\omega_p\,\hat{k}$$

$$\frac{d\vec{L}}{dt} = MR^2\omega_o\left(\left[\cos\theta\cos\alpha\frac{d\alpha}{dt} - \sin\theta\sin\alpha\frac{d\theta}{dt}\right]\hat{i} + \left[\sin\theta\cos\alpha\frac{d\alpha}{dt} + \cos\theta\sin\alpha\frac{d\theta}{dt}\right]\hat{j} - \sin\alpha\frac{d\alpha}{dt}\hat{k}\right)$$
$$+ Ml^2\left[\sin^2\alpha\frac{d\omega_p}{dt} + 2\sin\alpha\cos\alpha\frac{d\alpha}{dt}\right]\hat{k}$$

$$\vec{N} = \vec{l}\times M\vec{g} \qquad \vec{l} = l(\cos\theta\sin\alpha\,\hat{i} + \sin\theta\sin\alpha\,\hat{j} + \cos\alpha\,\hat{k})$$

$$= Ml\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \cos\theta\sin\alpha & \sin\theta\sin\alpha & \cos\alpha \\ 0 & 0 & -g \end{vmatrix}$$

$$= Ml(-g\sin\theta\sin\alpha\,\hat{i} + g\cos\theta\sin\alpha\,\hat{j})$$

$\frac{d\vec{L}}{dt} = \vec{N}$, for ω_o large, nutation is small $\rightarrow \frac{d\alpha}{dt} \cong 0$, and $\frac{d\theta}{dt} = \omega_p$

Equating components:

$$R^2\omega_0[-\sin\theta\sin\alpha\,\omega_p] = -g\ell\sin\theta\sin\alpha$$

$$R^2\omega_0[\cos\theta\sin\alpha\,\omega_p] = g\ell\cos\theta\sin\alpha$$

Each equation gives

$$\omega_p = \frac{g\ell}{R^2\omega_0}$$

when $\frac{d\alpha}{dt} \cong 0$

4-13

$L_0 = L_{sp0} + L_{p0}$

$L_{sp0} = M r_0^2 \omega_{y0}$, ω_{y0} = angular velocity of present year

$= M r_0^2 \, 2\pi / Y_0$, Y_0 is present year

$L_{p0} = \frac{2}{5} M a^2 \omega_0$

$L_1 = L_{sp1} + L_{p1}$

$L_{sp1} = M r^2 \omega_y$

$L_{p1} = \frac{2}{5} M a^2 \omega$

$L_0 = L_1$

$$r_0^2 \omega_{y0} + \frac{2}{5} a^2 \omega_0 = r^2 \omega_y + \frac{2}{5} a^2 \omega$$

$$\frac{2}{5} a^2(\omega_0 - \omega) + r_0^2 \omega_{y0} = r^2 \omega_y$$

$$= r \, V_{tan}$$

$$\frac{2}{5} \frac{a^2(\omega_0 - \omega)}{r_0^2 \omega_{y0}} + 1 = \frac{r \, V_{tan}}{r_0 V_{tan0}}$$

$$\frac{Y_0}{5\pi} \frac{a^2}{r_0^2}(\omega_0 - \omega) + 1 = \frac{r\sqrt{\frac{MG}{r}}}{r_0 \sqrt{\frac{MG}{r_0}}}$$

$$= \frac{r^{1/2}}{r_0^{1/2}}$$

$$r^{1/2} = r_0^{1/2}\left(1 + \frac{a^2}{r_0^2}(\omega_0 - \omega)\frac{Y_0}{5\pi}\right)$$

$$\boxed{r = r_0 \left(1 + \frac{a^2}{r_0^2}(\omega_0 - \omega)\frac{Y_0}{5\pi}\right)^2}$$

$F = ma$

$= \frac{mM}{r^2} G$

$a = \frac{MG}{r^2}$

$\frac{V^2}{r} = \frac{MG}{r^2}$

$V_{tan} = \sqrt{\frac{MG}{r}}$

$a = 3960$ mi

$r_0 = 9.29 \times 10^7$ mi

$\omega_0 = \frac{2\pi}{24 hr} = 7.27 \times 10^{-5} /s$

$Y_0 = 3.15 \times 10^7 s$

$\omega_{y0} = \frac{2\pi}{Y_0}$

$$r = r_0\left(1 + \frac{1}{5\pi}\frac{a^2}{r_0^2}(\omega_0 - \omega_{y_0})Y_0\right)^2$$

$$= r_0\left(1 + \frac{1}{5\pi}\frac{a^2}{r_0^2}(\omega_0 Y_0 - 2\pi)\right)^2$$

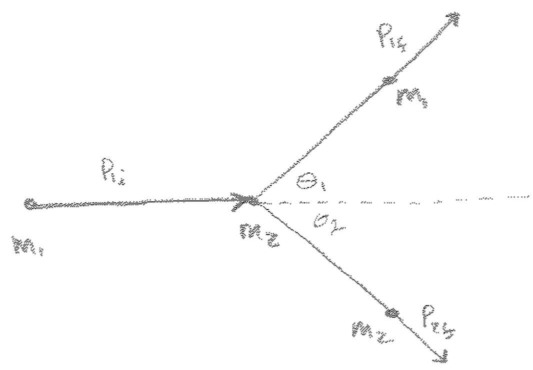
$$= (9.29\times10^7\,mi)\left(1 + \frac{1}{5\pi}\left(\frac{3960}{9.29\times10^7}\right)^2\left((7.27\times10^{-5})(3.15\times10^7) - 2\pi\right)\right)^2$$

$r = 92900049\,mi$

$\boxed{r \text{ increases } 49 \text{ miles}}$

If the moon was taken into account the change in distance from the sun would be less due to tidal friction from the moon.

4-15



$$P_{1i} = P_{1f}\cos\theta_1 + P_{2f}\cos\theta_2 \qquad P_{1f}\sin\theta_1 = P_{2f}\sin\theta_2$$

$$P_{1i} - P_{2f}\cos\theta_2 = P_{1f}\cos\theta_1 \qquad P_{1f}^2\sin^2\theta_1 = P_{2f}^2\sin^2\theta_2$$

$$P_{1i}^2 - 2P_{1i}P_{2f}\cos\theta_2 + P_{2f}^2\cos^2\theta_2 = P_{1f}^2\cos^2\theta_1$$

$$P_{2f}^2\sin^2\theta_2 = P_{1f}^2\sin^2\theta_1$$

$$P_{1i}^2 - 2P_{i}P_{2f}\cos\theta_2 + P_{2f}^2 = P_{1f}^2$$

$$\frac{P_{1i}^2}{2m_1} = \frac{P_{1f}^2}{2m_1} + \frac{P_{2f}^2}{2m_2}$$

$$\frac{P_{1i}^2 - P_{1f}^2}{m_1} = \frac{P_{2f}^2}{m_2}$$

$$\frac{P_{1i}^2 - \left(P_{1i}^2 - 2P_{1i}P_{2f}\cos\theta_2 + P_{2f}^2\right)}{m_1} = \frac{P_{2f}^2}{m_2}$$

$$\frac{2P_{1i}P_{2f}\cos\theta_2 - P_{2f}^2}{m_1} = \frac{P_{2f}^2}{m_2}$$

$$P_{2f}^2\left(\frac{1}{m_2} + \frac{1}{m_1}\right) = \frac{2P_{1i}P_{2f}\cos\theta_2}{m_1}$$

$$P_{2f}\left[\frac{m_1+m_2}{m_1 m_2}P_{2f} - \frac{2P_{1i}\cos\theta_2}{m_1}\right] = 0$$

$$P_{2f} = \frac{m_1 m_2}{m_1+m_2} \quad \frac{2P_{1i}\cos\theta_2}{m_1}$$

$$P_{2f} = \frac{2m_2\cos\theta_2\, P_{1i}}{m_1+m_2}$$

$$T_{2f} = \frac{P_{2f}^2}{2m_2}$$

$$= \frac{4\left(\frac{m_2}{m_1+m_2}\right)^2 \cos^2\theta_2 P_{1i}^2}{2m_2}$$

$$\boxed{T_{2f} = 2\frac{m_2}{(m_1+m_2)^2}\cos^2\theta_2 P_{1i}^2}$$

T_{2f} is a max at $\theta_2 = 0$ since $\cos^2\theta_2 = 1$ at $\theta_2 = 0$ provided m_1 does not miss m_2

When T_{2f} is a max

$$T_{1i} = T_{1f} + T_{2f}$$

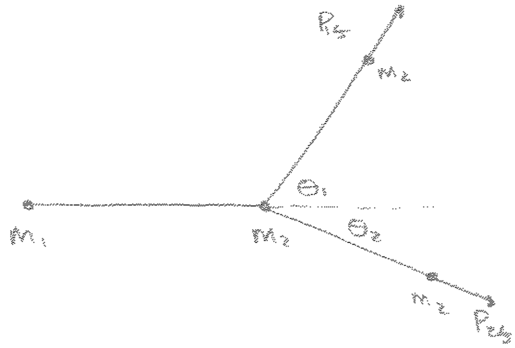
$$T_{1i} - T_{1f} = T_{2f}$$

$$= \frac{2m_2}{(m_1+m_2)^2} P_{1i}^2 \qquad \frac{P_{1i}^2}{2m_1} = T_{1i}$$

$$= \frac{2m_2}{(m_1+m_2)} 2m_1 T_{1i}$$

$$\boxed{T_{1i} - T_{1f} = \frac{4m_1m_2}{(m_1+m_2)^2} T_{1i}}$$

4-17



$$P_{1i} = P_{1f}\cos\theta_1 + P_{2f}\cos\theta_2 \qquad P_{1f}\sin\theta_1 = P_{2f}\sin\theta_2 \qquad \frac{P_{1i}^2}{m_1} - \frac{P_{1f}^2}{m_1} = \frac{P_{2f}^2}{m_2}$$

$$P_{1i} - P_{1f}\cos\theta_1 = P_{2f}\cos\theta_2 \qquad P_{2f}^2 = m_2\left(\frac{P_{1i}^2 - P_{1f}^2}{m_1}\right)$$

$$P_{1i}^2 - 2P_{1i}P_{1f}\cos\theta_1 + P_{1f}^2\cos^2\theta_1 = P_{2f}^2\cos^2\theta_2$$

$$P_{1f}^2\sin^2\theta_1 = P_{2f}^2\sin^2\theta_2$$

$$P_{1i}^2 + P_{1f}^2 - 2P_{1i}P_{1f}\cos\theta_1 = P_{2f}^2$$

$$= m_2\left(\frac{P_{1i}^2 - P_{1f}^2}{m_1}\right)$$

$$m_2 = \frac{m_1}{P_{1i}^2 - P_{1f}^2}\left(P_{1i}^2 + P_{1f}^2 - 2P_{1i}P_{1f}\cos\theta_1\right)$$

$$= \frac{m_1}{P_{1i}^2 - P_{1f}^2}P_{1i}^2\left(1 + \frac{P_{1f}^2}{P_{1i}^2} - 2\frac{P_{1f}}{P_{1i}}\cos\theta_1\right)$$

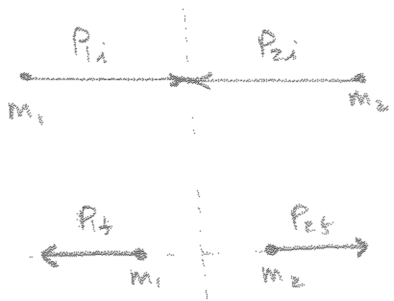
$$= \frac{m_1}{\frac{P_{1i}^2 - P_{1f}^2}{P_{1i}^2}}\left(1 + \frac{P_{1f}^2}{P_{1i}^2} - 2\frac{P_{1f}}{P_{1i}}\cos\theta_1\right)$$

$$= \frac{m_1}{1 - \frac{P_{1f}^2}{P_{1i}^2}}\left(1 + \frac{P_{1f}^2}{P_{1i}^2} - 2\frac{P_{1f}}{P_{1i}}\cos\theta_1\right)$$

$$\boxed{m_2 = \frac{m_1}{1-\alpha}\left(1 + \alpha^2 - 2\alpha\cos\theta_1\right), \quad \alpha = \frac{P_{1f}}{P_{1i}}}$$

To verify the elastic collision measure θ_2 and confirm.

4-19



$$P_{1i} - P_{2i} = -P_{1f} + P_{2f} \qquad \frac{P_{1i}^2}{m_1} + \frac{P_{2i}^2}{m_2} = \frac{P_{1f}^2}{m_1} + \frac{P_{2f}^2}{m_2}$$

$$m_1V_{1i} - m_2V_{2i} = -m_1V_{1f} + m_2V_{2f} \qquad m_1V_{1i}^2 + m_2V_{2i}^2 = m_1V_{1f}^2 + m_2V_{2f}^2$$

$$m_1V_{1i} + m_1V_{1f} = m_2V_{2i} + m_2V_{2f} \qquad m_1(V_{1i}^2 - V_{1f}^2) = m_2(V_{2f}^2 - V_{2i}^2)$$

$$m_1(V_{1i} + V_{1f}) = m_2(V_{2i} + V_{2f}) \qquad m_1(V_{1i} - V_{1f})(V_{1i} + V_{1f}) = m_2(V_{2f} - V_{2i})(V_{2f} + V_{2i})$$

$$\frac{m_1(V_{1i} - V_{1f})(V_{1i} + V_{1f}) = m_2(V_{2f} - V_{2i})(V_{2f} + V_{2i})}{m_1(V_{1i} + V_{1f}) \qquad = m_2(V_{2i} + V_{2f})}$$

$$V_{1i} - V_{1f} = V_{2f} - V_{2i}$$

$$V_{2f} + V_{1f} = V_{1i} + V_{2i}$$

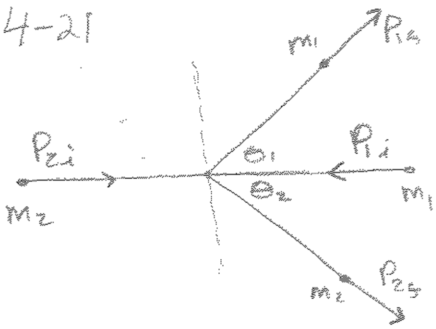
$$V_{2f} - (-V_{1f}) = V_{1i} - (-V_{2i})$$

comparing to 4.85

$$V_{2F} - V_{1F} = e(V_{1I} - V_{2I})$$

e must equal 1

4-21



Given: $m_1, m_2, P_{1i}, P_{2i}, \theta_1$

Find: P_{1f}

$$P_{2i} - P_{1i} = P_{1f}\cos\theta_1 + P_{2f}\cos\theta_2\,, \quad P_{1f}\sin\theta_1 = P_{2f}\sin\theta_2\,, \quad \frac{P_{1i}^2}{m_1} + \frac{P_{2i}^2}{m_2} = \frac{P_{1f}^2}{m_1} + \frac{P_{2f}^2}{m_2}$$

$$(P_{2i} - P_{1i}) - P_{1f}\cos\theta_1 = P_{2f}\cos\theta_2$$

$$(P_{2i} - P_{1i})^2 - 2(P_{2i} - P_{1i})P_{1f}\cos\theta_1 + P_{1f}^2\cos^2\theta_1 = P_{2f}^2\cos^2\theta_2$$

$$P_{1f}^2\sin^2\theta_1 = P_{2f}^2\sin^2\theta_2$$

$$(P_{2i} - P_{1i})^2 + P_{1f}^2 - 2(P_{2i} - P_{1i})P_{1f}\cos\theta_1 = P_{2f}^2\,, \quad \frac{P_{2f}^2}{m_2} = \frac{P_{1i}^2 - P_{1f}^2}{m_1} + \frac{P_{2i}^2}{m_2}$$

$$P_{2f}^2 = \frac{m_2}{m_1}(P_{1i}^2 - P_{1f}^2) + P_{2i}^2$$

$$(P_{2i} - P_{1i})^2 + P_{1f}^2 - 2(P_{2i} - P_{1i})P_{1f}\cos\theta_1 = \frac{m_2}{m_1}(P_{1i}^2 - P_{1f}^2) + P_{2i}^2$$

$$= \frac{m_2}{m_1}P_{1i}^2 - \frac{m_2}{m_1}P_{1f}^2 + P_{2i}^2$$

$$P_{1f}^2 + \frac{m_1}{m_2}P_{1f}^2 - 2(P_{2i} - P_{1i})\cos\theta_1 P_{1f} = \frac{m_2}{m_1}P_{1i}^2 + P_{2i}^2 - (P_{2i} - P_{1i})^2$$

$$\left(1 + \frac{m_1}{m_2}\right)P_{1f}^2 - 2(P_{2i} - P_{1i})\cos\theta_1 P_{1f} = \frac{m_2}{m_1}P_{1i}^2 + P_{2i}^2 - (P_{2i} - P_{1i})^2$$

$$P_{1f}^2 - \frac{2(P_{2i} - P_{1i})}{1+\gamma}\cos\theta_1 P_{1f} + \frac{(P_{2i} - P_{1i})^2\cos^2\theta_1}{(1+\gamma)^2} = \frac{\gamma P_{1i}^2 + P_{2i}^2 - (P_{2i} - P_{1i})^2}{1+\gamma} + \frac{(P_{2i} - P_{1i})^2\cos^2\theta_1}{(1+\gamma)^2}$$

$$\left(P_{1f} - \frac{(P_{2i} - P_{1i})}{1+\gamma}\cos\theta_1\right)^2 = \frac{(1+\gamma)\left[\gamma P_{1i}^2 + P_{2i}^2\right] - (1+\gamma)(P_{2i} - P_{1i})^2 + (P_{2i} - P_{1i})^2\cos^2\theta_1}{(1+\gamma)^2} \qquad \gamma = \frac{m_2}{m_1}$$

$$= \left[\gamma P_{1i}^2 + P_{2i}^2 + \gamma^2 P_{1i}^2 + \gamma P_{2i}^2 - \gamma(P_{2i} - P_{1i})^2 - (P_{2i} - P_{1i})^2 + (P_{2i} - P_{1i})^2\cos^2\theta_1\right]/(1+\gamma)^2$$

$$= \left[\gamma P_{1i}^2 + P_{2i}^2 + \gamma^2 P_{1i}^2 + \gamma P_{2i}^2 - \gamma\left[P_{2i}^2 - 2P_{2i}P_{1i} + P_{1i}^2\right] - (P_{2i} - P_{1i})^2(1 - \cos^2\theta)\right]/(1+\gamma)^2$$

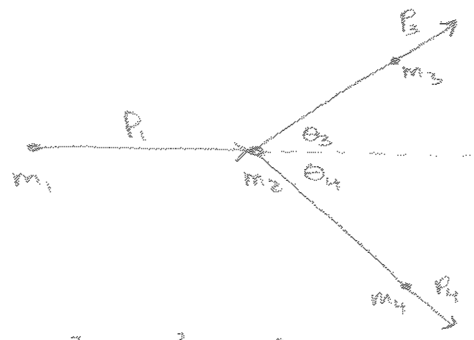
$$= \left[\gamma P_{1i}^2 + P_{2i}^2 + \gamma^2 P_{1i}^2 + \gamma P_{2i}^2 - \gamma P_{2i}^2 + 2\gamma P_{2i}P_{1i} - \gamma P_{1i}^2 - (P_{2i} - P_{1i})^2\sin^2\theta\right]/(1+\gamma)^2$$

$$= \left[\gamma^2 P_{1i}^2 + 2\gamma P_{2i}P_{1i} + P_{2i}^2 - (P_{2i} - P_{1i})^2\sin^2\theta\right]/(1+\gamma)^2$$

$$= \left[(\gamma P_{1i} + P_{2i})^2 - (P_{2i} - P_{1i})^2\sin^2\theta\right]/(1+\gamma)^2$$

$$\boxed{(1+\gamma)P_{1f} = (P_{2i} - P_{1i})\cos\theta_1 \pm \left[(\gamma P_{1i} + P_{2i})^2 - (P_{2i} - P_{1i})^2\sin^2\theta\right]^{1/2}}$$

4-23



$$Q + \frac{P_1^2}{2m_1} = \frac{P_3^2}{2m_3} + \frac{P_4^2}{2m_4}$$

$$P_1 = P_3\cos\theta_3 + P_4\cos\theta_4 \qquad P_3\sin\theta_3 = P_4\sin\theta_4$$

$$P_1 = P_3\cos\theta_3 + \frac{P_3\sin\theta_3\cos\theta_4}{\sin\theta_4}$$

$$= P_3\left(\frac{\cos\theta_3\sin\theta_4 + \sin\theta_3\cos\theta_4}{\sin\theta_4}\right)$$

$$P_3 = \frac{P_1\sin\theta_4}{\sin(\theta_3+\theta_4)} \qquad \sin(A+B) = \cos A\sin B + \sin A\cos B$$

$$\frac{P_4\sin\theta_4}{\sin\theta_3} = \frac{P_1\sin\theta_4}{\sin(\theta_3+\theta_4)}$$

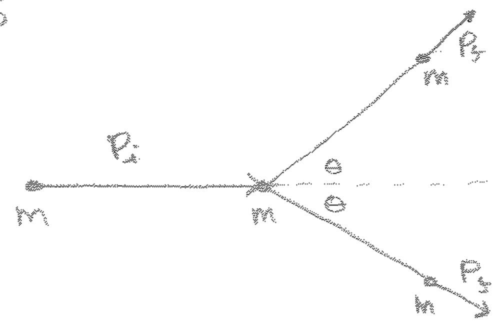
$$P_4 = \frac{P_1\sin\theta_3}{\sin(\theta_3+\theta_4)}$$

$$Q = \frac{P_3^2}{2m_3} + \frac{P_4^2}{2m_4} - \frac{P_1^2}{2m_1}$$

$$= \frac{P_1^2\sin^2\theta_4}{2m_3\sin^2(\theta_3+\theta_4)} + \frac{P_1^2\sin^2\theta_3}{2m_4\sin^2(\theta_3+\theta_4)} - \frac{P_1^2}{2m_1}$$

$$\boxed{Q = \frac{P_1^2}{2m_1}\left[\frac{(m_1/m_3)\sin^2\theta_4 + (m_1/m_4)\sin^2\theta_3}{\sin^2(\theta_3+\theta_4)} - 1\right]}$$

4-25



$$P_i = P_f\cos\theta + P_f\cos\theta \qquad P_f\sin\theta = P_f\sin\theta$$

$$P_i = 2P_f\cos\theta$$

$$\frac{P_i^2}{2m} = \frac{P_f^2}{2m} + \frac{P_f^2}{2m} + Q$$

$$= \frac{P_f^2}{m} + Q$$

$$= \frac{P_i^2}{4m\cos^2\theta} + Q$$

$$Q = \frac{P_i^2}{2m} - \frac{P_i^2}{4m\cos^2\theta}$$

$$\boxed{Q = \frac{P_i^2}{2m}\left(1 - \frac{1}{2}\cos^{-2}\theta\right)}$$

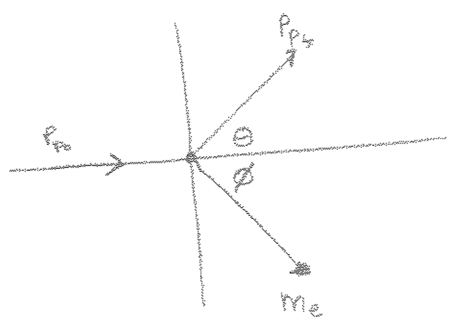
4-27

$$T_p = \frac{hc}{\lambda} \qquad P_p = \frac{h}{\lambda}$$

$$T_e = mc^2\left(\left[1-\frac{v^2}{c^2}\right]^{-1/2}-1\right)$$

Conservation of momentum

$$\vec{P}_{p_0} = \vec{P}_{p_f} + \vec{P}_{e_f} \quad \{P_{e_0}=0\}$$



$$P_{p_0} = P_{p_f}\cos\theta + P_{e_f}\cos\phi \qquad P_{p_f}\sin\theta = P_{e_f}\sin\phi$$

$$P_{p_f}^2\sin^2\theta = P_{e_f}^2\sin^2\phi$$

$$P_{p_0} - P_{p_f}\cos\theta = P_{e_f}\cos\phi$$

$$P_{p_0}^2 - 2P_{p_0}P_{p_f}\cos\theta + P_{p_f}^2\cos^2\theta = P_{e_f}^2\cos^2\phi$$

$$P_{p_f}^2\sin^2\theta = P_{e_f}^2\sin^2\phi$$

$$P_{p_0}^2 + P_{p_f}^2 - 2P_{p_0}P_{p_f}\cos\theta = P_{e_f}^2 \quad (1)$$

Conservation of energy

$$T_{p_0} + T_{e_0} = T_{p_f} + T_{e_f}$$

Expressing T is terms of P gives

$$T_p = \frac{hc}{\lambda}, \quad P_p = \frac{h}{\lambda}$$

$$P_p c = \frac{hc}{\lambda} = T_p$$

$$\frac{P_{e_f}^2}{2m} = T_{e_f} + \frac{T_{e_f}^2}{2mc^2} \quad \{Eq. 4.75\}$$

$$P_{e_f}^2 c^2 = 2mc^2 T_{e_f} + T_{e_f}^2$$

$$T_{e_f}^2 + 2mc^2 T_{e_f} = P_{e_f}^2 c^2$$

$$T_{e_f}^2 + 2mc^2 T_{e_f} + (mc^2)^2 = P_{e_f}^2 c^2 + (mc^2)^2$$

$$(T_{e_f} + mc^2)^2 = P_{e_f}^2 c^2 + (mc^2)^2$$

$$T_{e_f} = \left(P_{e_f}^2 c^2 + (mc^2)^2\right)^{1/2} - mc^2$$

$$T_{e_o} = 0$$

$$T_{p_o} = T_{p_f} + \left(P_{e_f}^2 c^2 + (mc^2)^2\right)^{1/2} - mc^2$$

$$P_{p_o} c - P_{p_f} c + mc^2 = \left(P_{e_f}^2 c^2 + (mc^2)^2\right)^{1/2}$$

$$c(P_{p_o} - P_{p_f}) + mc^2 = \left(P_{e_f}^2 c^2 + (mc^2)^2\right)^{1/2}$$

$$c^2(P_{p_o} - P_{p_f})^2 + 2c(P_{p_o} - P_{p_f})mc^2 + (mc^2)^2 = P_{e_f}^2 c^2 + (mc^2)^2$$

$$(P_{p_o} - P_{p_f})^2 + \frac{2mc^2(P_{p_o} - P_{p_f})}{c} = P_{e_f}^2 \quad (2)$$

Eliminating $P_{e_f}^2$ by using Eqs. (1) and (2) gives

$$P_{p_o}^2 + P_{p_f}^2 - 2P_{p_o}P_{p_f}\cos\theta = P_{p_o}^2 + P_{p_f}^2 - 2P_{p_o}P_{p_f} + \frac{2mc^2(P_{p_o} - P_{p_f})}{c}$$

$$-2P_{p_o}P_{p_f}\cos\theta = -2P_{p_o}P_{p_f} + \frac{2mc^2(P_{p_o} - P_{p_f})}{c}$$

$$P_{p_o}P_{p_f}(1-\cos\theta) = \frac{mc^2(P_{p_o} - P_{p_f})}{c}$$

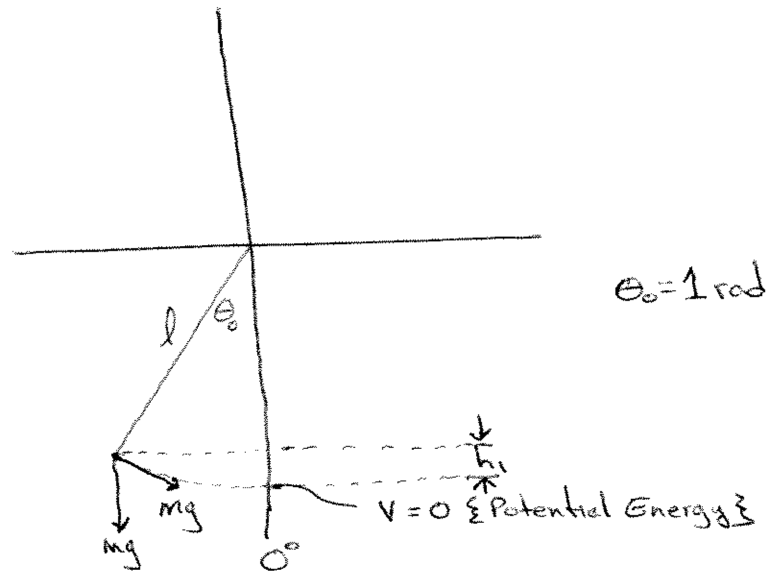
$$\frac{h}{\lambda_o}\frac{h}{\lambda_f}(1-\cos\theta) = \frac{mc^2\left(\frac{h}{\lambda_o} - \frac{h}{\lambda_f}\right)}{c}$$

$$= \frac{mc^2 h\left(\frac{1}{\lambda_o} - \frac{1}{\lambda_f}\right)}{c}$$

$$\frac{h}{mc}(1-\cos\theta) = \lambda_o\lambda_f\left(\frac{1}{\lambda_o} - \frac{1}{\lambda_f}\right)$$

$$\boxed{\frac{h}{mc}(1-\cos\theta) = \lambda_f - \lambda_o}$$

5-11



a) Total Energy at $\theta = 0° = 0$ rad

① Potential: $mgh_1 = mgl(1-\cos\theta_0)$

② work done by $F = mg$ along arc from θ_0 to 0

$$W = -\int mg\,dr$$

$$= -\int_{\theta_0}^{0} mg\,l\,d\theta$$

$$= +mgl\theta_0$$

③ $E_T = mgh_1 + W$

$$= mgl(1-\cos\theta_0) + mgl\theta_0$$

$$= mgl(1-\cos\theta_0 + \theta_0) \quad \{\theta_0 = 1\text{ rad}\}$$

$$E_T = mgl(2-\cos(1))$$

when $\dot{\theta} = 0$ all energy is potential

$$PE = E_T$$

$$mgl(1-\cos\theta) = mgl(2-\cos(1))$$

$$1-\cos\theta = 2-\cos(1)$$

$$\cos\theta = \cos(1) - 1$$

$$\cos\theta = -.46$$

$$\boxed{\theta = 2.05\text{ rad} = 117°}$$

b) $I\ddot{\theta} = -mgl\sin\theta - mgl \quad , \quad \theta_0 < \theta < 0$

$ml^2\ddot{\theta} = -mgl\sin\theta - mgl$

$\ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{g}{l}$

$\ddot{\theta} + \frac{g}{l}\sin\theta = -\frac{g}{l}$

$\ddot{\theta} + \frac{g}{l}\theta = -\frac{g}{l} \quad \{\text{assume } \sin\theta \doteq \theta \text{ for } \theta < 1\}$

$\theta_h = A\cos\omega_0 t + B\sin\omega_0 t \quad , \quad \omega_0 = \left(\frac{g}{l}\right)^{1/2}$

let $\theta_p = A$

$\frac{g}{l}A = -\frac{g}{l}$

$A = -1$

$\theta(t) = A\cos\omega_0 t + B\sin\omega_0 t - 1$

$\theta(0) = \theta_0 = 1 = A - 1$

$1 = A - 1$

$A = 2$

$\dot{\theta}(t) = -A\omega_0\sin\omega_0 t + B\omega_0\cos\omega_0 t$

$\dot{\theta}(0) = 0 = B\omega_0$

$B = 0$

$\theta = 2\cos\omega_0 t - 1$

$0 = 2\cos\omega_0 t - 1$

$\cos\omega_0 t = \frac{1}{2}$

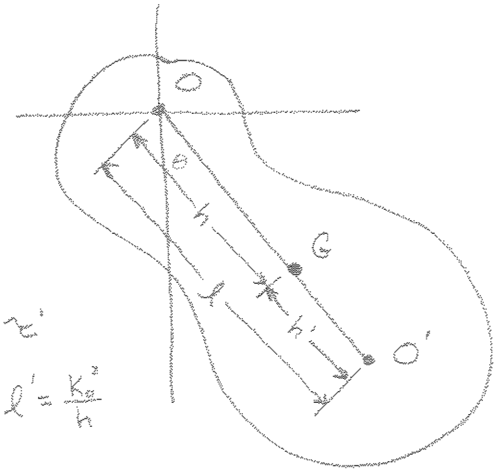
$\omega_0 t = \cos^{-1}\left(\frac{1}{2}\right)$

$\omega_0 t = \frac{\pi}{3}$

$t = \frac{\pi}{3}\omega_0^{-1}$

$$\boxed{t = \frac{\pi}{3}\left(\frac{l}{g}\right)^{1/2}}$$

5-13



$\tau = \tau'$

$l = l' = \frac{K_o^2}{h}$

$K_o^2 = K_G^2 + h^2$ {∥ axis Theorem}

a) $\omega = \left(\frac{g}{l}\right)^{1/2}$

$\omega^2 = \frac{g}{l}$

$\frac{4\pi^2}{\tau^2} = \frac{g}{l}$

$\tau^2 = 4\pi^2 \frac{l}{g}$

$\tau^2 = \frac{4\pi^2}{g}\frac{K_o^2}{h}$

$\tau^2 = \frac{4\pi^2}{g}\frac{(K_G^2 + h^2)}{h}$

$hh' = K_G^2$

$\tau^2 = \frac{4\pi^2}{g}\frac{(hh' + h^2)}{h}$

$= \frac{4\pi^2}{g}(h' + h)$

$\boxed{g = \frac{4\pi^2 (h' + h)}{\tau^2}}$

b) $\tau' = \tau(1+\delta)$, $\delta \ll 1$

$l \neq l'$

$K_o^2 = K_G^2 + h^2$, $K_{o'}^2 = K_G^2 + h'^2$

$l = \frac{K_o^2}{h}$, $l' = \frac{K_{o'}^2}{h'}$

$\tau^2 = \frac{4\pi^2}{g}\frac{K_o^2}{h}$

$= \frac{4\pi^2}{g}\frac{(K_G^2 + h^2)}{h}$

$K_G^2 = \frac{\tau^2 g h}{4\pi^2} - h^2$

$\tau'^2 = \frac{4\pi^2}{g}\frac{K_{o'}^2}{h'}$

$\tau'^2 = \frac{4\pi^2}{g}\frac{(K_G^2 + h'^2)}{h'}$

$= \frac{4\pi^2}{g}\left[\frac{\left(\frac{\tau^2 g h}{4\pi^2} - h^2\right) + h'^2}{h'}\right]$

$= \frac{\tau^2 g h}{g h'} - \frac{4\pi^2 h^2}{g h'} + \frac{4\pi^2 h'^2}{g h'}$

$\tau'^2 = \tau^2(1+\delta)^2$

$\approx \tau^2(1+2\delta)$

$\tau^2(1+2\delta)h' = \frac{\tau^2}{g} g h - \frac{4\pi^2}{g}(h^2 - h'^2)$

$\tau^2[(1+2\delta)h' - h] = -\frac{4\pi^2}{g}(h+h')(h-h')$

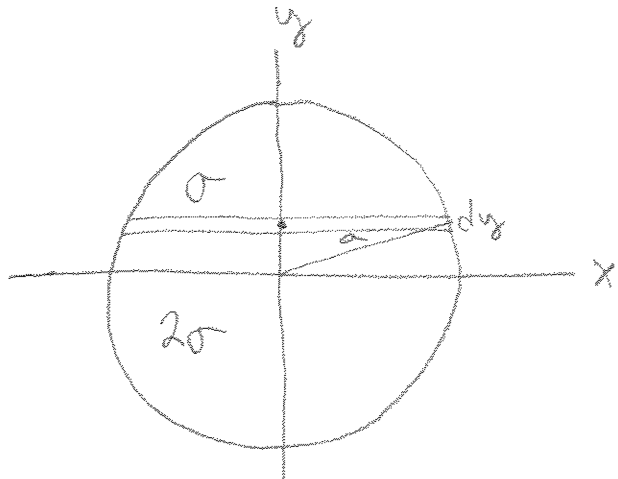
$g[h' + 2h'\delta - h] = -\frac{4\pi^2}{\tau^2}(h+h')(h-h')$

$g(h' + 2h'\delta) - gh = -g(h-h')$

$(h'-h) + 2h'\delta = h' - h$

$1 + \frac{2h'\delta}{h'-h} = 1 \rightarrow$ Correction factor is $\boxed{1 + \frac{2h'\delta}{h'-h}}$

15. A circular disk of radius a lies in the xy-plane with its center at the origin. The half of the disk above the x-axis has a density σ per unit area, and the half below the x-axis has a density 2σ. Find the center of mass G, and the moments of inertia about the x-, y-, and z-axes, and about parallel axes through G. Make as much use of labor-saving theorems as possible.



$$x^2+y^2=a^2$$
$$x=\sqrt{a^2-y^2}$$
$$l=2x$$
$$=2\sqrt{a^2-y^2}$$

$$\boxed{\bar{X}=0}$$

$$\bar{y}=\frac{\int \tilde{y}\,dm}{\int dm} \qquad \tilde{y}=y$$

$$dm_1=\sigma\,da \qquad dm_2=2\sigma\,da$$
$$=\sigma x\,dy \qquad =2\sigma x\,dy$$
$$=2\sigma(a^2-y^2)^{1/2}dy \qquad =4\sigma(a^2-y^2)^{1/2}dy$$

$$=\frac{\int_0^a y\,dm_1-\int_0^{-a} y\,dm_2}{M_1+M_2}$$

$$M_1=\frac{\sigma\pi a^2}{2} \qquad M_2=\sigma\pi a^2$$

$$=\frac{2\sigma\int_0^a y(a^2-y^2)^{1/2}dy-4\sigma\int_0^{-a} y(a^2-y^2)^{1/2}dy}{\frac{3}{2}\sigma\pi a^2}$$

$$u=a^2-y^2$$
$$du=-2y\,dy$$
$$-du=2y\,dy$$

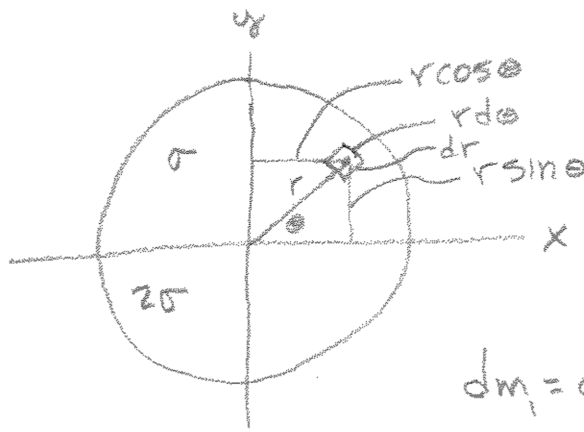
$$=\frac{-\int u^{1/2}du+2\int u^{1/2}du}{\frac{3}{2}\pi a^2}$$

$$=\frac{-\frac{2}{3}(a^2-y^2)^{3/2}\Big|_0^a+\frac{4}{3}(a^2-y^2)^{3/2}\Big|_0^{-a}}{\frac{3}{2}\pi a^2}$$

$$\bar{y} = \frac{\frac{2}{3}a^3 - \frac{4}{3}a^3}{\frac{3}{2}\pi a^2}$$

$$= \frac{-\frac{2}{3}a^3}{\frac{3}{2}\pi a^2}$$

$$\boxed{\bar{y} = \frac{-4a}{9\pi}}$$



$$dm_1 = \sigma\, da \qquad dm_2 = 2\sigma\, da$$
$$= \sigma\, r\, dr\, d\theta \qquad = 2\sigma\, r\, dr\, d\theta$$

$$I_x = \int r^2\, dm$$

$$= \int_0^{\pi}\int_0^{a} (r^2 \sin^2\theta)\, \sigma\, r\, dr\, d\theta + \int_{\pi}^{2\pi}\int_0^{a} (r^2\sin^2\theta)\, 2\sigma\, r\, dr\, d\theta$$

$$\sin^2\theta = \tfrac{1}{2}(1-\cos 2\theta)$$

$$= \sigma \int_0^{\pi}\int_0^{a} r^3\, dr \sin^2\theta\, d\theta + 2\sigma \int_{\pi}^{2\pi}\int_0^{a} r^3\, dr \sin^2\theta\, d\theta$$

$$= \sigma \frac{a^4}{4} \int_0^{\pi} \tfrac{1}{2}(1-\cos 2\theta)\, d\theta + \frac{\sigma a^4}{2} \int_{\pi}^{2\pi} \tfrac{1}{2}(1-\cos 2\theta)\, d\theta$$

$$= \frac{\sigma a^4}{8}\left[\theta\Big|_0^{\pi} - \tfrac{1}{2}\sin 2\theta\Big|_0^{\pi}\right] + \frac{\sigma a^4}{4}\left[\theta\Big|_0^{\pi} - \tfrac{1}{2}\sin 2\theta\Big|_0^{\pi}\right]$$

$$= \frac{\sigma a^4 \pi}{8} + \frac{\sigma a^4 \pi}{4}$$

$$\boxed{I_x = \frac{3\sigma a^4 \pi}{8}}$$

$$I_y = \int_0^{\pi}\int_0^a (r^2\cos^2\theta)\sigma r\,dr\,d\theta + \int_{\pi}^{2\pi}\int_0^a (r^2\cos^2\theta)2\sigma r\,dr\,d\theta$$

$$= \sigma\int_0^{\pi}\int_0^a r^3 dr \cos^2\theta\, d\theta + 2\sigma\int_{\pi}^{2\pi}\int_0^a r^3 dr\cos^2\theta\, d\theta \qquad \cos^2\theta = \tfrac{1}{2}(1+\cos 2\theta)$$

Since $\int_0^{\pi}\cos 2\theta\, d\theta = \int_{\pi}^{2\pi}\cos 2\theta = 0$

$$\boxed{I_y = I_x}$$

$I_z = I_x + I_y$ { $\perp$ axis theorem }

$= 2I_x$

$$\boxed{I_z = \frac{3\sigma a^4\pi}{4}}$$

Since $G = (0, \bar{y})$

$I_{Gy} = I_y$

$$\boxed{I_{Gy} = \frac{3\sigma a^4\pi}{8}}$$

// axis theorem: $I_x = G_{ox} + Mh^2$

$I_{Gx} = I_x - Mh^2 \quad h = \frac{-4a}{9\pi}$

$$= \frac{3\sigma a^4\pi}{8} - \frac{3}{2}\pi a^2\sigma\left(\frac{-4a}{9\pi}\right)^2$$

$$= \frac{3\sigma a^4\pi}{8} - \frac{3}{2}\pi a^2\sigma\left(\frac{16a^4}{81\pi}\right)$$

$$\boxed{I_{Gx} = \frac{3\sigma a^4\pi}{8} - \frac{8\sigma a^4}{27\pi} = (81\pi^2 - 64)\sigma a^4/216\pi}$$

$$I_{Gz} = I_{Gx} + I_{Gy} \quad \{\perp \text{ axis theorem}\}$$

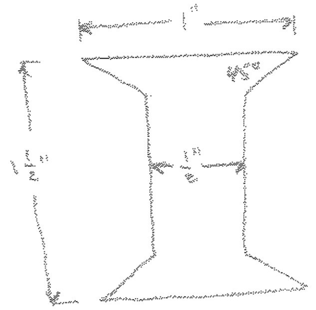
$$= \frac{(81\pi^2 - 64)\sigma a^4}{216\pi} + \frac{3\sigma a^4 \pi}{8}$$

$$= \frac{(81\pi^2 - 64)\sigma a^4}{216\pi} + \frac{81\sigma a^4 \pi^2}{216\pi}$$

$$= \frac{2(81\pi^2 - 32)\sigma a^4}{216\pi}$$

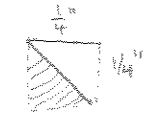
$$\boxed{I_{Gz} = \frac{(81\pi^2 - 32)\sigma a^4}{108\pi}}$$

5-19

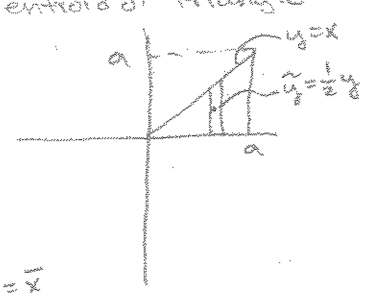


$d_t = 0.03"$

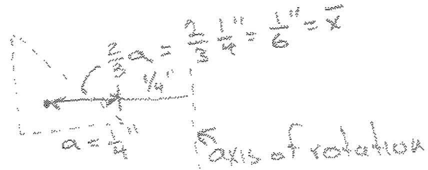
Theorem of Pappus applied to triangle section



Centroid of triangle



$$\bar{y} = \frac{1}{M}\int \tilde{y}\,dm$$
$$= \frac{1}{M}\int \tilde{y}\rho\,dA$$
$$= \frac{\rho}{M}\int \tilde{y}y\,dx \quad , \; M = \rho A$$
$$= \frac{1}{A}\int \frac{1}{2}y\,y\,dx$$
$$= \frac{1}{2A}\int y^2 dx$$
$$= \frac{1}{2A}\int_0^a x^2 dx$$
$$= \frac{1}{2A}\frac{x^3}{3}\Big|_0^a$$
$$= \frac{1}{2A}\frac{a^3}{3} \qquad A = \frac{1}{2}b\cdot h = \frac{1}{2}a^2$$
$$= \frac{1}{2(\frac{1}{2}a^2)}\frac{a^3}{3}$$
$$\bar{y} = \frac{a}{3}$$



$$r = \frac{1}{6}" + \frac{1}{4}"$$
$$r = \frac{5}{12}"$$

$$V_\triangle = A\pi(2r)$$
$$= \frac{1}{2}\left(\frac{1}{4}\right)^2\pi(2)\left(\frac{5}{12}"\right)$$
$$= \frac{1}{2}\cdot\frac{1}{16}\cdot\pi\cdot\frac{5}{6}$$
$$= \frac{5\pi}{192}\text{ in.}^3$$

$$V_\square = A\pi(2r)$$
$$A = (1")\left(\frac{1}{4}"\right) = \frac{1}{4}\text{ in}^2$$
$$r = \frac{1}{4}" + \frac{1}{8}" \quad \left\{\bar{x} = \frac{1}{8}"\right\}$$
$$V_\square = \frac{1}{4}\pi(2)\left(\frac{3}{8}\right)$$
$$V_\square = \frac{3\pi}{16}\text{ in.}^3$$

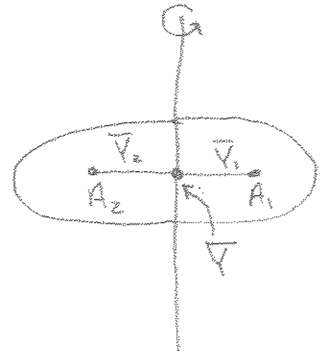
$$V = 2V_\triangle + V_\square$$
$$= \left(\frac{5\pi}{96} + \frac{3\pi}{16}\right)\text{in.}^3$$
$$V = \frac{23\pi}{96}\text{ in.}^3$$

Assuming 100% packing density {thread is deformable}:

$$V = A\cdot l$$
$$l = \frac{V}{A} \qquad A = \pi\left(\frac{d_t}{2}\right)^2$$
$$= \frac{23\pi}{96}\cdot\frac{1}{\pi\left(\frac{d_t}{2}\right)^2}$$
$$= \frac{23}{96}\cdot\frac{4}{d_t^2}$$
$$= \frac{23}{24}\cdot\frac{1}{(.03)^2}$$
$$\boxed{l = 1064.8\text{ in} = 29.6\text{ yd.} = 30\text{ yd.}}$$

5-21

Take the case of the axis of rotation passing through the axis of symmetry of a regular object



Theorem of Pappus #2

Entire object: $V = A\,2\pi\bar{Y}$

$V = A\,2\pi(0)$

$V = 0$

Left side of object

$V_2 = A_2 2\pi\bar{Y}_2$

Right side of object

$V_1 = A_1 2\pi\bar{Y}_1$

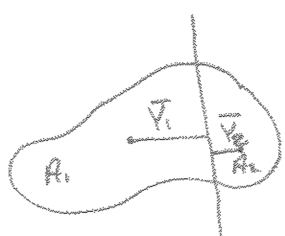
Since the volume generated using $\bar{Y}$ equals 0 then

$V = V_1 + V_2$

$0 = V_1 + V_2$

$-V_1 = V_2$

Therefore one volume is considered a negative volume. This negative volume can be viewed as a volume that erases the positive volume generated. So, in general

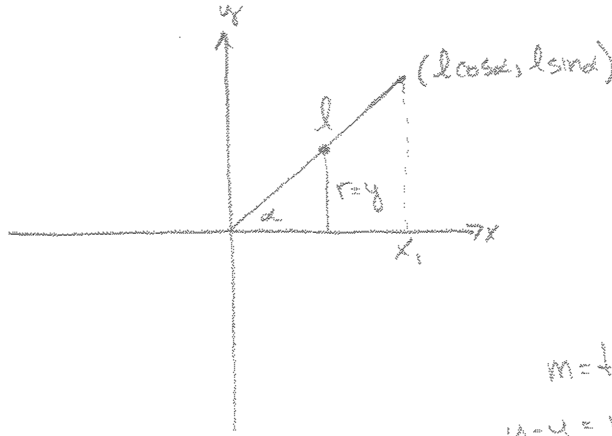


$V = |V_1 - V_2|$

$V = |2\pi(A_1\bar{Y}_1 - A_2\bar{Y}_2)|$

5-23

a)



$dm = \rho\, ds$

$ds^2 = dx^2 + dy^2$

$m = \tan\alpha$

$y - y_1 = m(x - x_1)$

$y - l\sin\alpha = \tan\alpha(x - l\cos\alpha)$

$y - l\sin\alpha = x\tan\alpha - l\sin\alpha$

$y = x\tan\alpha$

$ds^2 = dx^2 + dx^2\tan^2\alpha$

$= dx^2(1+\tan^2\alpha)$

$ds = dx\sec\alpha$

$$I_x = \int r^2 dm$$

$$= \int_0^{x_1} y^2 dm$$

$$= \int_0^{x_1} (x\tan\alpha)^2 \rho\, ds$$

$$= \rho\tan^2\alpha \int_0^{x_1} x^2 dx \sec\alpha$$

$$= \rho\tan^2\alpha\sec\alpha \int_0^{x_1} x^2 dx$$

$$= \rho\tan^2\alpha\sec\alpha \left.\frac{x^3}{3}\right|_0^{x_1} \qquad x_1 = l\cos\alpha$$

$$= \rho\tan^2\alpha\sec\alpha \frac{(l\cos\alpha)^3}{3}$$

$$= \frac{\rho l^3}{3} \frac{\sin^2\alpha}{\cos^2\alpha} \frac{1}{\cos\alpha} \frac{\cos^3\alpha}{1}$$

$$I_x = \frac{\rho l^3}{3}\sin^2\alpha$$

$M K^2 = I_x \qquad M = \rho l$

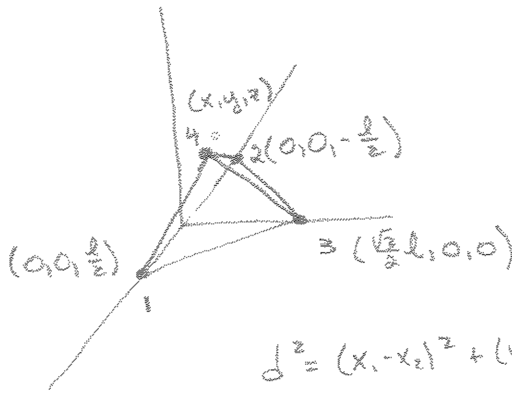
$$\rho l K^2 = \frac{\rho l^3}{3}\sin^2\alpha$$

$$K^2 = \frac{l^2\sin^2\alpha}{3}$$

$$\boxed{K = \frac{l\sin\alpha}{\sqrt{3}}}$$

b)

1) Determine vertices coordinates



$$d^2 = (x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2$$

1) $d_{4-1}^2 = l^2 = x^2 + y^2 + (z-\frac{l}{2})^2$

2) $d_{4-2} = l^2 = x^2 + y^2 + (z+\frac{l}{2})^2$

3) $d_{4-3} = l^2 = (x - \frac{\sqrt{3}}{2}l)^2 + y^2 + z^2$

Eq. 1) = Eq. 2)

$$x^2+y^2+z^2-lz+\frac{l^2}{4} = x^2+y^2+z^2+lz+\frac{l^2}{4}$$

$$0 = 2lz$$

$$\boxed{z=0}$$

Eq. 1) = Eq. 3) ; $z=0$

$$x^2+y^2+\frac{l^2}{4} = x^2 - \sqrt{3}lx + \frac{3}{4}l^2 + y^2$$

$$\frac{l^2}{4} = -\sqrt{3}lx + \frac{3}{4}l^2$$

$$\frac{l^2}{2} = \sqrt{3}lx$$

$$\boxed{x = \frac{\sqrt{3}l}{6}}$$

Eq. 3) ; $z=0$, $x = \frac{\sqrt{3}l}{6}$

$$l^2 = \left(\frac{\sqrt{3}}{6}l - \frac{\sqrt{3}}{2}l\right)^2 + y^2$$

$$= \left(\frac{\sqrt{3}l}{6} - \frac{3\sqrt{3}}{6}l\right)^2 + y^2$$

$$= \left(-\frac{2\sqrt{3}l}{6}\right)^2 + y^2$$

$$= \frac{12l^2}{36} + y^2$$

$$\frac{36-12}{36}l^2 = y^2 \Rightarrow \frac{24}{36}l^2 = y^2 \Rightarrow y = \frac{2\sqrt{6}}{6}l \Rightarrow \boxed{\frac{\sqrt{6}}{3}l = y} \quad pt.4\left(\frac{\sqrt{3}}{6}l, \frac{\sqrt{6}}{3}l, 0\right)$$

The midpoint of each rod (centroid) is:

$R_{12} = (0,0,0) \quad R_{13} = \left(\frac{\sqrt{3}}{4}l, 0, \frac{l}{4}\right) \quad R_{23} = \left(\frac{\sqrt{3}}{4}l, 0, -\frac{l}{4}\right)$

$R_{14} = \left(\frac{\sqrt{3}}{12}l, \frac{\sqrt{6}}{6}l, \frac{l}{4}\right) \quad R_{24} = \left(\frac{\sqrt{3}}{12}l, \frac{\sqrt{6}}{6}l, -\frac{l}{4}\right) \quad R_{34} = \left(\frac{\sqrt{3}}{3}l, \frac{\sqrt{6}}{6}l, 0\right)$

$\boxed{\bar{z} = 0}$ due to symmetry.

$$\bar{x} = \frac{1}{M}\sum_{k=1}^{6} M_k \bar{x}_k = \frac{1}{S}\sum_{k=1}^{6} s_k \bar{x}_k$$

$$= \frac{1}{6l}\left[\frac{\sqrt{3}}{4}l^2 + \frac{\sqrt{3}}{4}l^2 + \frac{\sqrt{3}}{12}l^2 + \frac{\sqrt{3}}{12}l^2 + \frac{\sqrt{3}}{3}l^2\right]$$

$$= \frac{l}{6}\left[\frac{2\sqrt{3}}{4} + \frac{2\sqrt{3}}{12} + \frac{\sqrt{3}}{3}\right]$$

$$= \frac{l}{6}\left[\frac{6\sqrt{3} + 2\sqrt{3} + 4\sqrt{3}}{12}\right]$$

$$= \frac{l}{6}\left[\sqrt{3}\right]$$

$$\boxed{\bar{x} = \frac{\sqrt{3}l}{6}}$$

$$\bar{y} = \frac{l}{S}\sum_{k=1}^{6} s_k \bar{y}_k$$

$$= \frac{l}{6}\left[\frac{\sqrt{6}}{6} + \frac{\sqrt{6}}{6} + \frac{\sqrt{6}}{6}\right]$$

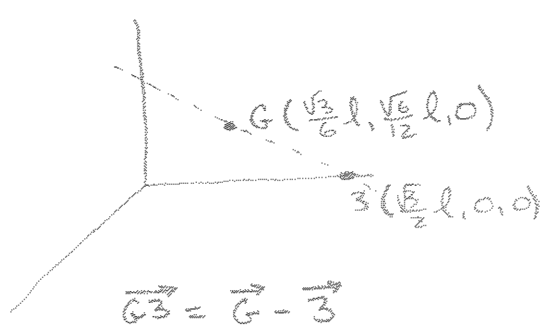
$$= \frac{l}{6}\left[\frac{\sqrt{6}}{2}\right]$$

$$\boxed{\bar{y} = \frac{\sqrt{6}l}{12}}$$

$$\boxed{\begin{array}{l}\text{Centroid of object:} \\ \bar{x}, \bar{y}, \bar{z} = \frac{\sqrt{3}l}{6}, \frac{\sqrt{6}}{12}l, 0\end{array}}$$

Axis of Rotation:

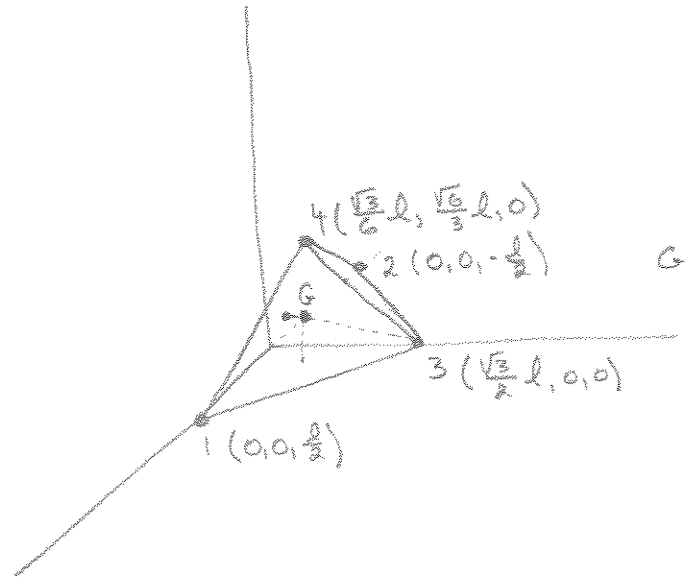
For convenience choose vertex 3:



$$\vec{G3} = \vec{G} - \vec{3}$$

$$= \left(\frac{\sqrt{3}}{6}l - \frac{\sqrt{3}}{2}l\right)\hat{x} + \frac{\sqrt{6}}{12}l\,\hat{y}$$

$$\vec{G3} = -\frac{\sqrt{3}}{3}l\,\hat{x} + \frac{\sqrt{6}}{12}l\,\hat{y}$$



$G\left(\frac{\sqrt{3}}{6}l, \frac{\sqrt{6}}{12}l, 0\right)$

$\vec{13} = \vec{1} - \vec{3} = -\frac{\sqrt{3}}{2}l\hat{x} + \frac{l}{2}\hat{z}$; $\vec{23} = \vec{2} - \vec{3} = -\frac{\sqrt{3}}{2}l\hat{x} - \frac{l}{2}\hat{z}$; $\vec{43} = \vec{4} - \vec{3} = \left(\frac{\sqrt{3}}{6} - \frac{\sqrt{3}}{2}\right)l\hat{x} + \frac{\sqrt{6}}{3}l\hat{y} = -\frac{\sqrt{3}}{3}l\hat{x} + \frac{\sqrt{6}}{3}l\hat{y}$

Axis of rotation: $\vec{G3} = -\frac{\sqrt{3}}{3}l\hat{x} + \frac{\sqrt{6}}{12}l\hat{y}$

$$\cos\alpha = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$$

$$|\vec{G3}| = \left(\frac{3}{9} + \frac{6}{144}\right)^{1/2} l = \left(\frac{1}{3} + \frac{1}{24}\right)^{1/2} l = \left(\frac{9}{24}\right)^{1/2} l$$

$$|\vec{G3}| = \frac{3l}{\sqrt{24}} = \frac{3\sqrt{24}\,l}{24} = \frac{2\sqrt{6}\,l}{8} = \frac{\sqrt{6}}{4}l$$

$$\cos\alpha_1 = \frac{\vec{G3}\cdot\vec{13}}{|\vec{G3}||\vec{13}|} = \frac{-\frac{\sqrt{3}}{3}\cdot -\frac{\sqrt{3}}{2}l^2 + 0 + 0}{\frac{\sqrt{6}}{4}l^2} = \frac{3}{6}\cdot\frac{4}{\sqrt{6}} = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$$

$$\cos^2\alpha_1 = \frac{6}{9} = \frac{2}{3}$$

$$\sin^2\alpha_1 = 1 - \cos^2\alpha_1 = 1 - \frac{2}{3}$$

$$\sin^2\alpha_1 = \frac{1}{3}$$

$$\sin\alpha_1 = \frac{\sqrt{3}}{3}$$

$$\cos\alpha_2 = \frac{\vec{G3}\cdot\vec{23}}{\frac{\sqrt{6}}{4}l^2} = \cos\alpha_1$$

$$\sin\alpha_2 = \sin\alpha_1 = \frac{\sqrt{3}}{3}$$

$$\cos\alpha_3 = \frac{\vec{G3}\cdot\vec{43}}{\frac{\sqrt{6}}{4}l^2} = \frac{\frac{3}{9} + \frac{6}{36}}{\frac{\sqrt{6}}{4}} = \frac{\frac{1}{3} + \frac{1}{6}}{\frac{\sqrt{6}}{4}} = \frac{1}{2}\cdot\frac{4}{\sqrt{6}} = \frac{2}{\sqrt{6}} = \frac{\sqrt{6}}{3}$$

$$\boxed{\sin\alpha_1 = \sin\alpha_3 = \sin\alpha_1 = \frac{\sqrt{3}}{3}}$$

Determine angle between axis of rotation and $\vec{41}$

$$\cos\alpha = \frac{\vec{G3}\cdot\vec{41}}{\frac{\sqrt{6}}{4}\ell^2}$$

$$\vec{41} = \vec{4} - \vec{1} = \frac{\sqrt{3}}{6}\ell\hat{x} + \frac{\sqrt{6}}{3}\ell\hat{y} - \frac{\ell}{2}\hat{z}$$

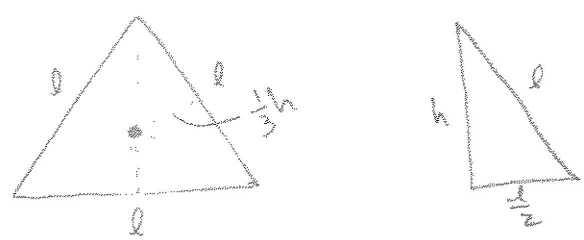
$$= \frac{\left(-\frac{\sqrt{3}}{3}\ell\right)\left(\frac{\sqrt{3}}{6}\ell\right) + \left(\frac{\sqrt{6}}{12}\ell\right)\left(\frac{\sqrt{6}}{3}\ell\right) + 0}{\frac{\sqrt{6}}{4}\ell^2}$$

$$= \frac{-\frac{3}{18} + \frac{6}{36}}{\frac{\sqrt{6}}{4}}$$

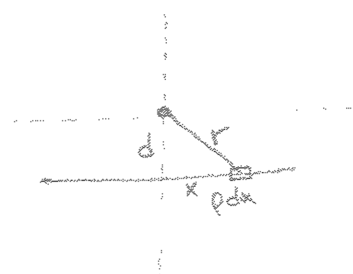
$$= \frac{-\frac{3}{18} + \frac{3}{18}}{\frac{\sqrt{6}}{4}}$$

$$= 0$$

Therefore, $\vec{G3} \perp \vec{41}$, $\vec{G3} \perp \vec{12}$, $\vec{G3} \perp \vec{42}$



$$h^2 = \ell^2 - \frac{\ell^2}{4} = \frac{3\ell^2}{4}$$
$$h = \frac{\ell\sqrt{3}}{2}$$



$$d = \frac{1}{3}h = \frac{\ell\sqrt{3}}{6}$$

$r^2 = x^2 + d^2 \qquad dm = \rho dx$

$$I = \int r^2 dm$$
$$= \rho\int_{-\ell/2}^{\ell/2}(x^2+d^2)dx$$
$$= \rho\left[\int_{-\ell/2}^{\ell/2} x^2 dx + d^2\int_{-\ell/2}^{\ell/2} dx\right]$$
$$= \rho\left[\frac{x^3}{3}\Big|_{-\ell/2}^{\ell/2} + d^2 x\Big|_{-\ell/2}^{\ell/2}\right]$$
$$= \rho\left[\frac{\ell^3}{24} + \frac{\ell^3}{24} + \frac{d^2\ell}{2} + \frac{d^2\ell}{2}\right]$$
$$= \rho\left[\frac{\ell^3}{12} + d^2\ell\right]$$

$$I = \frac{\rho\ell^3}{12} + \rho\left(\frac{\ell\sqrt{3}}{6}\right)^2\ell$$
$$= \frac{\rho\ell^3}{12} + \frac{\rho\ell^3 3}{36}$$
$$= \frac{\rho\ell^3}{12} + \frac{\rho\ell^3}{12}$$
$$= \frac{\rho\ell^3}{6}$$

Each rod is oriented the same with respect to the axis of rotation.

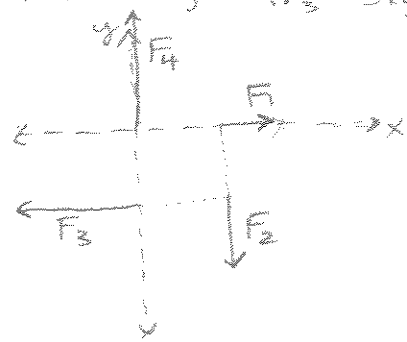
$$I_\Delta = 3I$$
$$= 3\,\frac{\rho l^3}{6}$$
$$= \frac{\rho l^3}{2}$$

$$I_{Pyr.} = I_\Delta + 3I_\alpha$$
$$= \frac{\rho l^3}{2} + \frac{3\rho l^3}{3}\sin^2\alpha$$
$$= \rho l^3\left[\frac{1}{2} + \sin^2\alpha\right] \qquad \sin^2\alpha = \sin^2\alpha_1$$
$$= \rho l^3\left[\frac{1}{2} + \frac{1}{3}\right] \qquad = \left(\frac{\sqrt{3}}{3}\right)^2$$
$$\qquad = \frac{1}{3}$$
$$= \rho l^3\left(\frac{5}{6}\right)$$

$$\boxed{I_{Pyr.} = \frac{5}{6}Ml^2}$$

6-26

$F_1 = 1\,\text{Kg-wt}$, $F_2 = 2\,\text{Kg-wt}$, $F_3 = 3\,\text{Kg-wt}$, $F_4 = 4\,\text{Kg-wt}$



$$\vec{F} = \sum_i F_i$$

$$(\vec{r} - \vec{r}_0) \times \vec{F} = \sum_i (\vec{r}_i - \vec{r}_0) \times F_i$$

$$\vec{F} = \sum_i F_i \; ; \; F_x = F_1 - F_3 \qquad F_y = -F_2 + F_4$$

$$= 1 - 3 \qquad\qquad = -2 + 4$$

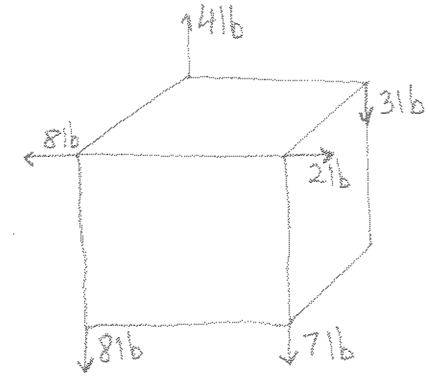
$$= -2 \qquad\qquad = 2$$

$$\vec{F} = -2\hat{x} + 2\hat{y}$$

$-\vec{F}$ = equilbrant

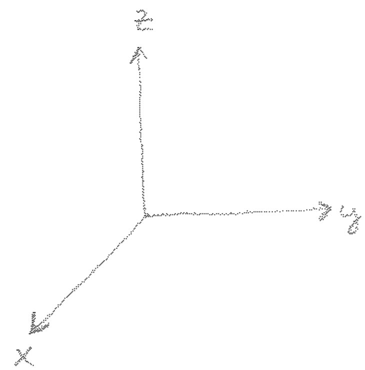
$$\boxed{-\vec{F} = \vec{F}_{eq} = 2\hat{x} - 2\hat{y}}$$

5-27



1 ft × 1 ft × 1 ft cube

a) Coordinate system at center of cube



$F_1 = 4\,\text{lb}$

$F_2 = 3\,\text{lb}$

$F_3 = 7\,\text{lb}$

$F_4 = 8\,\text{lb}$ (down)

$F_5 = 8\,\text{lb}$

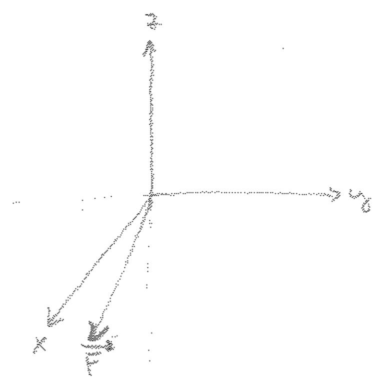
$F_6 = 2\,\text{lb}$

$\vec{F}_1 = 4\hat{z} \quad \vec{F}_2 = -3\hat{z} \quad \vec{F}_3 = -7\hat{z} \quad \vec{F}_4 = -8\hat{z} \quad \vec{F}_5 = -8\hat{y} \quad \vec{F}_6 = 2\hat{y}$

$$\vec{F}_o = \sum_i \vec{F}_i$$

$$= 0\hat{x} + (-8+2)\hat{y} + (4-3-7-8)\hat{z}$$

$$\boxed{\vec{F}_o = 0\hat{x} - 6\hat{y} - 14\hat{z}}$$



Torque at center of cube

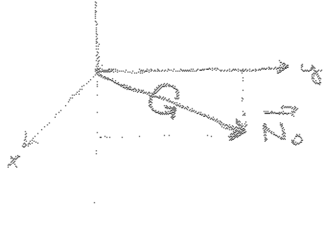
$$\vec{N}_o = \sum_i \vec{N}_{oi}$$

$$= \sum_i \vec{r}_{io} \times \vec{F}_i$$

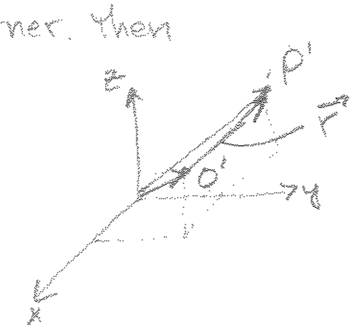
$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 4 \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -3 \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -7 \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -8 \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & -8 & 0 \end{vmatrix} + \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 2 & 0 \end{vmatrix}$$

$$= -2\hat{x} + 2\hat{y} \quad + -\tfrac{3}{2}\hat{x} - \tfrac{3}{2}\hat{y} + -\tfrac{7}{2}\hat{x} + \tfrac{7}{2}\hat{y} + 4\hat{x} + 4\hat{y} + 4\hat{x} - 4\hat{z} + -\hat{x} + \hat{z}$$

$$\vec{N}_o = 0\hat{x} + 8\hat{y} - 3\hat{z}$$



Let O' be the right front corner and P' be the adjacent rear corner. Then



$$\vec{r} = \vec{P'} - \vec{O'}$$
$$= -\tfrac{1}{2}\hat{x} + \tfrac{1}{2}\hat{y} + \tfrac{1}{2}\hat{z} - \left(\tfrac{1}{2}\hat{x} + \tfrac{1}{2}\hat{y} + \tfrac{1}{2}\hat{z}\right)$$
$$= -\hat{x}$$

$\vec{N_o} = \vec{r} \times \vec{F_c}$ where F_c acts at P'

$$= -\hat{x} \times F_c$$

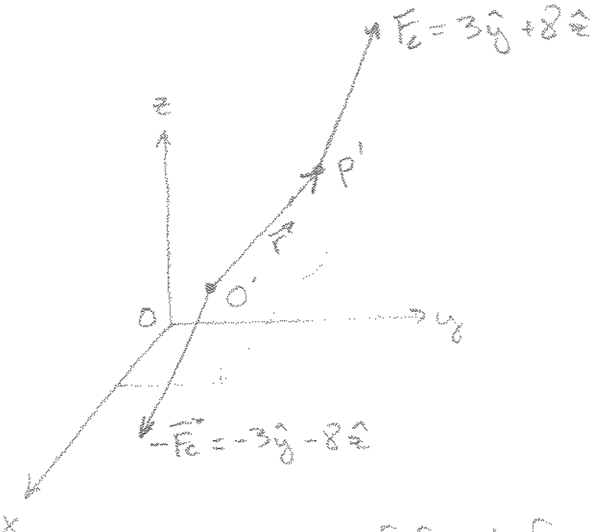
$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & 0 & 0 \\ F_x & F_y & F_z \end{vmatrix}$$

$$= F_z\hat{y} - F_y\hat{z}$$

$$8\hat{y} - 3\hat{z} = F_z\hat{y} - F_y\hat{z}$$

$$F_z = 8 \quad F_y = 3 \quad F_x = 0$$

$$\boxed{\begin{aligned} F_{c_{P'}} &= 0\hat{x} + 3\hat{y} + 8\hat{z} \\ -F_{c_{O'}} &= 0\hat{x} - 3\hat{y} - 8\hat{z} \end{aligned}}$$



b) Compute the equivalent couple about O and center of front face P''

$$\vec{r} = P'' - O$$
$$= \tfrac{1}{2}\hat{x}$$

$\vec{N_o} = \vec{r} \times F_c$ where F_c acts at P''

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \tfrac{1}{2} & 0 & 0 \\ F_x & F_y & F_z \end{vmatrix}$$

$$= -\frac{F_z}{2}\hat{y} + \frac{F_y}{2}\hat{z}$$

$$8\hat{y} - 3\hat{z} = -\frac{F_z}{2}\hat{y} + \frac{F_y}{2}\hat{z}$$

$$F_z = -16 \quad F_y = -6$$

$$F_{c_{p''}} = 0\hat{x} - 6\hat{y} - 16\hat{z}$$

$$-\vec{F_{c_o}} = 0\hat{x} + 6\hat{y} + 16\hat{z}$$

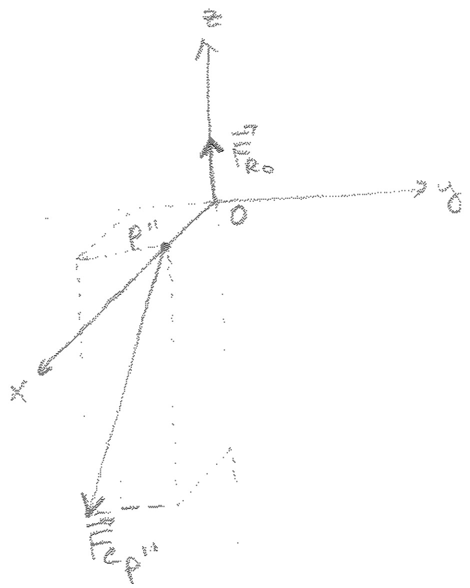
$-F_{c_o}$ acts within the y-z plane through the cube center O

$$\vec{F_{R_o}} = \vec{F} + (-\vec{F_{c_o}})$$

$$= 0\hat{x} - 6\hat{y} - 14\hat{z} + 0\hat{x} + 6\hat{y} + 16\hat{z}$$

$$\vec{F_{R_o}} = 0\hat{x} + 0\hat{y} + 2\hat{z}$$

$$\boxed{\begin{aligned} F_{R_o} &= 0\hat{x} + 0\hat{y} + 2\hat{z} \\ F_{c_{p''}} &= 0\hat{x} - 6\hat{y} - 16\hat{z} \end{aligned}}$$



c) The force $\vec{F}$ at O is

$$\vec{F_o} = 0\hat{x} - 6\hat{y} - 14\hat{z}$$

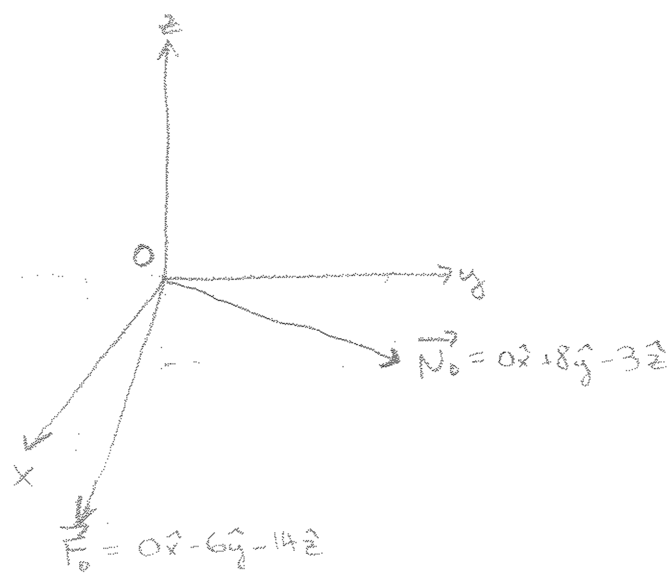
The torque $\vec{N}$ at O is

$$\vec{N_o} = 0\hat{x} + 8\hat{y} - 3\hat{z}$$

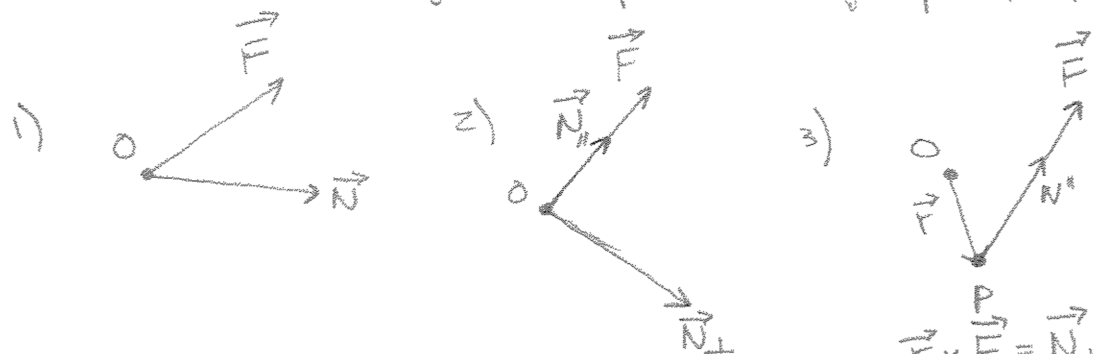
Let $\vec{r}$ be a vector $\perp$ to $\vec{N_o}$

$\vec{r} = a\hat{x}$ since $\hat{x}$ $\perp$ to y-z plane.

Determine the components of $\vec{N_o}$ parallel and normal to $\vec{F_o}$



Steps to find a single force plus a torque parallel to the force



$\vec{r} \times \vec{F} = \vec{N}_{\perp}$

$\vec{F}_0 = -6\hat{y} - 14\hat{z} \qquad \vec{N}_0 = 8\hat{y} - 3\hat{z}$

Component of $\vec{N}_0$ on $\vec{F}_0$

$$N_{\parallel} = |\vec{N}_0|\cos\theta = \frac{\vec{F}_0 \cdot \vec{N}_0}{|\vec{F}_0|}$$

$$= \frac{-48 + 42}{(36+196)^{1/2}}$$

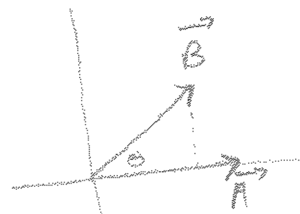
$$= \frac{-6}{\sqrt{232}}$$

$$\vec{N}_{\parallel} = N_{\parallel}\hat{F}_0$$

$$= \frac{-6}{\sqrt{232}} \frac{-6\hat{y} - 14\hat{z}}{\sqrt{232}}$$

$$= \frac{36}{232}\hat{y} + \frac{84}{232}\hat{z}$$

$$\boxed{\vec{N}_{\parallel} = \frac{9}{58}\hat{y} + \frac{21}{58}\hat{z}}$$



$$|\vec{A}||\vec{B}|\cos\theta = \vec{A} \cdot \vec{B}$$

$$|\vec{B}|\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}|} = B_{\parallel}$$

$$N_{\perp}^2 = N_0^2 - N_{\parallel}^2 \qquad N_0^2 = 64 + 9 = 73$$

$$N_{\parallel}^2 = \frac{81 + 441}{58^2} = \frac{522}{58^2}$$

$$= 73 - \frac{522}{58^2}$$

$$= \frac{73 \cdot 58^2 - 522}{58^2}$$

$$= \frac{73 \cdot 58^2 - 9 \cdot 58}{58^2}$$

$$N_{\perp} = \frac{\sqrt{58}\,(73 \cdot 58 - 9)^{1/2}}{58}$$

$\vec{r} = a\hat{x}$ since $\vec{F}_0$ and $\vec{N}_0$ are in y-z plane, and $a\hat{x} \times \vec{F}_0$ is in the $\vec{N}_{\perp}$ direction.

$$|\vec{r}| = \frac{|\vec{N}_{\perp}|}{|\vec{F}_0|} = \frac{N_{\perp}}{F_0}$$

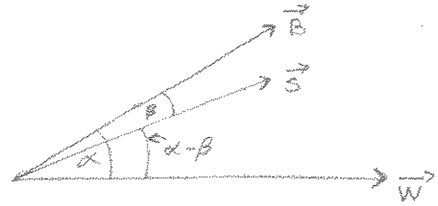
$$= \frac{\sqrt{58}\,(73 \cdot 58 - 9)^{1/2}}{58\sqrt{232}}$$

$$= \frac{(73 \cdot 58 - 9)^{1/2}}{58 \cdot 2}$$

$$|\vec{r}| = \frac{65}{116}$$

$$\boxed{\vec{r} = \frac{65}{116}\hat{x}}$$

5-29

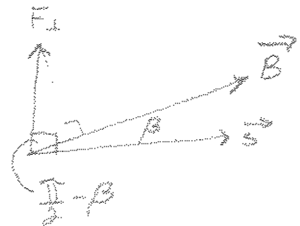


i) Component of $\vec{W} \perp$ to $\vec{s}$

$$W_\perp = |\vec{W}| \sin\theta$$
$$= W \sin(\alpha - \beta)$$

$$F_\perp = K W_\perp$$
$$= KW \sin(\alpha - \beta)$$

Component of $F_\perp \parallel$ to $\vec{B}$



$$F_{\parallel_B} = F_\perp \cos\left(\tfrac{\pi}{2} - \beta\right)$$

$$F_{\parallel_B} = KW \sin(\alpha - \beta) \cos\left(\tfrac{\pi}{2} - \beta\right)$$

$$= KW \sin(\alpha - \beta) \sin\beta \qquad \sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

$$= KW \sin\beta(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$$

$$= KW\left[\sin\alpha\sin\beta\cos\beta - \cos\alpha\sin^2\beta\right]$$

$$\frac{dF_{\parallel B}}{d\beta} = KW\left[\sin\alpha(-\sin^2\beta + \cos^2\beta) - 2\cos\alpha\sin\beta\cos\beta\right]$$

$$0 = \sin\alpha(\cos^2\beta - \sin^2\beta) - 2\cos\alpha\sin\beta\cos\beta$$

$$2\cos\alpha\sin\beta\cos\beta = \sin\alpha(\cos^2\beta - \sin^2\beta)$$

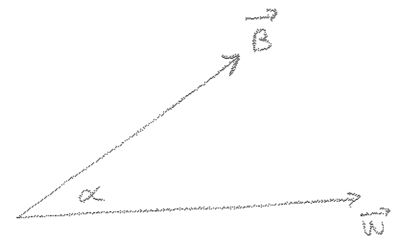
$$\frac{2\sin\beta\cos\beta}{\cos^2\beta - \sin^2\beta} = \tan\alpha$$

$$2\sin\beta\cos\beta = \sin 2\beta$$
$$\cos^2\beta = \tfrac{1}{2}(1 + \cos 2\beta)$$
$$\sin^2\beta = \tfrac{1}{2}(1 - \cos 2\beta)$$

$$\frac{\sin 2\beta}{\cos 2\beta} = \tan\alpha$$

$$2\beta = \alpha$$

$$\boxed{\beta = \tfrac{1}{2}\alpha}$$



ii) Component of $\vec{B}$ // to $\vec{W}$

$B_{//} = W\cos\alpha$

$\vec{f} = -K_1\vec{V}$ where $\vec{V}$ is boat velocity

$F_{//B} = K_2 W \sin(\alpha-\beta)\sin\beta$, $\beta = \frac{1}{2}\alpha$

Component of boat velocity // to wind

$V_{B_{//}} = V_B \cos\alpha$

At maximum velocity

$f = F_{//B}$

$K_1 V_B = K_2 W \sin(\alpha-\beta)\sin\beta$

$= K_2 W \sin(\alpha/2)\sin\alpha/2$, $\beta = \frac{1}{2}\alpha$

$= K_2 W \sin^2(\alpha/2)$

$V_B = \frac{K_2}{K_1} W \sin^2(\alpha/2)$

$V_{B_{//}} = \frac{K_2}{K_1} W \sin^2(\alpha/2)\cos\alpha$

$\frac{dV_{B_{//}}}{d\alpha} = \frac{K_2}{K_1} W \left[-\sin^2(\alpha/2)\sin\alpha + \cos\alpha\sin(\alpha/2)\cos(\alpha/2)\right]$

$0 = -\sin^2(\alpha/2)\sin\alpha + \cos\alpha\sin(\alpha/2)\cos(\alpha/2)$

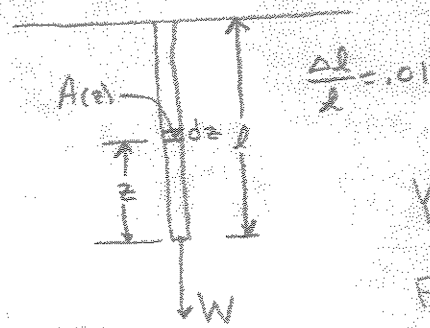
$\sin^2(\alpha/2)\sin\alpha = \cos\alpha\sin(\alpha/2)\cos(\alpha/2)$

$\frac{\sin(\alpha/2)}{\cos(\alpha/2)} = \frac{\cos\alpha}{\sin\alpha}$ $\qquad \tan 2u = \frac{2\tan u}{1-\tan^2 u}$

$\tan(\alpha/2) = \frac{1-\tan^2(\alpha/2)}{2\tan(\alpha/2)}$

$2\tan^2(\alpha/2) = 1-\tan^2(\alpha/2) \Rightarrow \tan(\alpha/2) = \frac{1}{\sqrt{3}} \Rightarrow \frac{\alpha}{2} = 30° \Rightarrow$ $\boxed{\alpha = 60°}$

5-31



$$Y = \frac{Fl}{A\Delta l}$$

F is force stretching the cable

The total force on the cable element is

$$F(z) = W + w\int_0^z A(z)\,dz$$

Using Young's modulus

$$Y = \frac{Fl}{A\Delta l}$$

$$A(z) = \frac{F(z)}{.01Y}$$

$$= \frac{100F(z)}{Y}$$

$$A(z) = \frac{100}{Y}\left[W + w\int_0^z A(z)\,dz\right]$$

$$\frac{dA}{dz} = \frac{100w}{Y}A$$

$$\frac{dA}{A} = \frac{100w}{Y}dz$$

$$\int_{A_0}^{A}\frac{dA}{A} = \int_0^z \frac{100w}{Y}dz$$

$$\ln A - \ln A_0 = \frac{100wz}{Y}$$

$$\ln\frac{A}{A_0} = \frac{100wz}{Y}$$

$$\frac{A}{A_0} = e^{\frac{100wz}{Y}}$$

$$A = A_0 e^{\frac{100wz}{Y}}$$

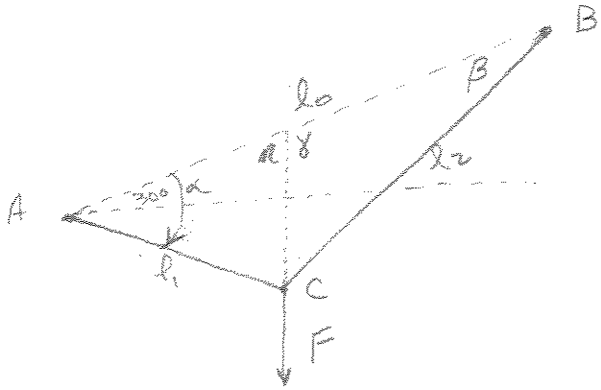
At $z=0$ the force is W

$$A_0 = A(0) = \frac{100F(0)}{Y}$$

$$A_0 = \frac{100W}{Y}$$

$$\boxed{A = \frac{100W}{Y}e^{\frac{100wz}{Y}}}$$

5-33



$l_0 = 15$
$l_1 = 8$
$l_2 = 12$ } $l_1 + l_2 = 20$
$F = 2000$
$\gamma = 120°$

Law of cosines:

$$l_2^2 = l_0^2 + l_1^2 - 2 l_0 l_1 \cos\alpha \qquad l_1^2 = l_0^2 + l_2^2 - 2 l_0 l_2 \cos\beta$$

$$\cos\alpha = \frac{l_0^2 + l_1^2 - l_2^2}{2 l_0 l_1} \qquad \cos\beta = \frac{l_0^2 + l_2^2 - l_1^2}{2 l_0 l_2}$$

$$= \frac{15^2 + 8^2 - 12^2}{2(15)(8)} \qquad = \frac{15^2 + 12^2 - 8^2}{2(15)(12)}$$

$$= \frac{145}{240} \qquad = \frac{305}{360}$$

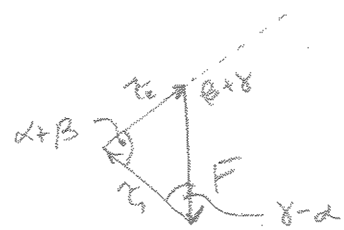
$$\alpha = \cos^{-1}\left(\frac{29}{48}\right) \qquad \beta = \cos^{-1}\left(\frac{61}{72}\right)$$

$$\alpha = 52.83° \qquad \beta = 32.09°$$

a) Position of C and cable tensions

$$\alpha - 30° = 52.83° - 30°$$
$$= 22.83°$$

$\boxed{\text{C is 8 ft from A and 22.83° below horizontal}}$



$\gamma = 120°$

$$T_1 = F\frac{\sin(\beta+\gamma)}{\sin(\alpha+\beta)} \qquad T_2 = F\frac{\sin(\gamma-\alpha)}{\sin(\alpha+\beta)}$$

$$= 2000\frac{\sin(152.09)}{\sin(84.92°)} \qquad = 2000\frac{\sin(67.17°)}{\sin(84.92°)}$$

$$\boxed{T_1 = 940\text{ lb} \qquad T_2 = 1851\text{ lb}}$$

b) $d = \frac{1}{2}''$, $Y = 5 \times 10^5 \frac{lb}{in^2}$

$l_1 = l_1'(1 + K\tau_1)$ $l_2 = l_2'(1 + K\tau_2)$ $\frac{1}{K} = YA = \pi(\frac{d}{2})^2 Y$

$K = \frac{4}{\pi d^2 Y}$

$= \frac{4}{\pi(\frac{1}{4})(5 \times 10^5)}$

$= \frac{16}{\pi(5 \times 10^5)}$

$= 1.02 \times 10^{-5}$

zero approximation from a)

$l_1 = 8 = l_1'$ $l_2 = 12 = l_2'$, $\tau_1 = 939$, $\tau_2 = 1850$

first approximation

$l_1 = 8(1 + (1.02 \times 10^{-5})(940))$
$= 8.077$

$l_2 = 12(1 + (1.02 \times 10^{-5})(1851))$
$= 12.226$

$\cos\alpha = \frac{15^2 + (8.077)^2 - (12.226)^2}{2(15)(8.077)}$

$= .5809$

$\alpha = 54.48°$

$\cos\beta = \frac{15^2 + (12.226)^2 - (8.077)^2}{2(15)(12.226)}$

$= .8431$

$\beta = 32.53°$

$\tau_1 = F \frac{\sin(\beta + \gamma)}{\sin(\alpha + \beta)}$

$= 2000 \frac{\sin(32.53 + 120°)}{\sin(54.48° + 32.53°)}$

$\tau_1 = 924$

$\tau_2 = F \frac{\sin(\gamma - \alpha)}{\sin(\alpha + \beta)}$

$= 2000 \frac{\sin(120° - 54.48°)}{\sin(54.48° + 32.53°)}$

$\tau_2 = 1823$

second approximation

$l_1 = 8.007(1 + (1.02 \times 10^{-5})(924))$
$= 8.082$

$l_2 = 12.226(1 + (1.02 \times 10^{-5})(1823))$
$= 12.460$

$\cos\alpha = \frac{15^2 + (8.082)^2 - (12.460)^2}{2(15)(8.082)}$

$\alpha = 56.15°$

$\cos\beta = \frac{15^2 + (12.460)^2 - (8.082)^2}{2(15)(12.460)}$

$\beta = 32.59°$

$\tau_1 = 2000 \frac{\sin(32.59° + 120°)}{\sin(56.15° + 32.59°)}$

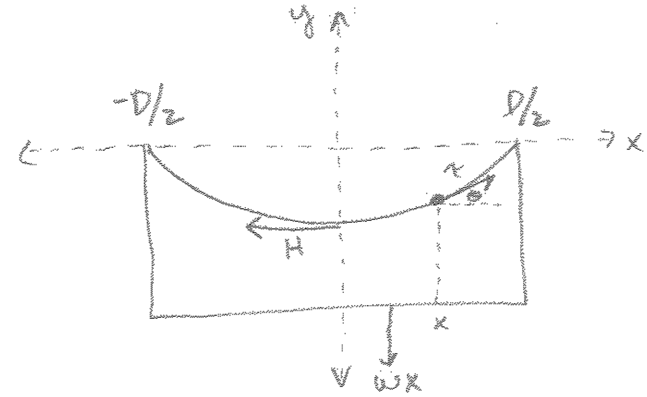
$\tau_2 = 2000 \frac{\sin(120° - 56.15°)}{\sin(56.15° + 32.59°)}$

$\boxed{\tau_1 = 921 \text{ lb}}$ $\boxed{\tau_2 = 1796 \text{ lb}}$

$\alpha - 30° = 26.15°$

$\boxed{\text{C is 8.082 ft from A and 26.15° below horizontal}}$

5-35



$$\tau \sin\theta = \omega x$$

$$\tau \cos\theta = H$$

$$\tan\theta = \frac{\omega}{H} x$$

$$\frac{dy}{dx} = \frac{\omega}{H} x$$

$$dy = \frac{\omega}{H} x \, dx$$

$$y = \frac{\omega}{2H} x^2 + \alpha$$

An endpoint is $(0, \frac{D}{2})$

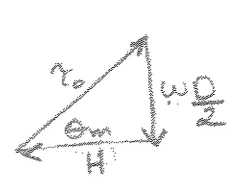
$$0 = \frac{\omega}{2H}\left(\frac{D}{2}\right)^2 + \alpha$$

$$\alpha = \frac{-\omega D^2}{8H}$$

$$y = \frac{\omega}{2H} x^2 - \frac{\omega D^2}{8H}$$

$$y = \frac{\omega}{2H}\left(x^2 - \frac{D^2}{4}\right)$$

The maximum tension, τ_0, will occur at maximum θ, the end of the cable.



$$\tau_0^2 = \frac{\omega^2 D^2}{4} + H^2$$

$$4H^2 = 4\tau_0^2 - \omega^2 D^2$$

$$H = \frac{(4\tau_0^2 - \omega^2 D^2)^{1/2}}{2}$$

$$y = \frac{\omega}{2\frac{(4\tau_0^2 - \omega^2 D^2)^{1/2}}{2}}\left(x^2 - \frac{D^2}{4}\right)$$

$$\boxed{y = \omega\left(x^2 - \frac{D^2}{4}\right)\left(4\tau_0^2 - \omega^2 D^2\right)^{-1/2}}$$

5-37

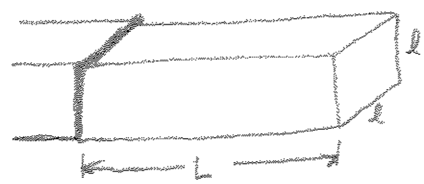
Eq (5.165) $\quad y = -\dfrac{L^3}{YK^2A}\left[\tfrac{1}{8}W + \tfrac{1}{3}W'\right] - \dfrac{L}{nA}\left[\tfrac{1}{2}W + W'\right]$

First term is deflection due to bending.

Second term is deflection due to shear

W' is force exerted at free end of beam

Given: $Y = n$, $W' = 0$, assume $W \ll Yl^2$



a) $\dfrac{L^3}{YK^2A}\left[\tfrac{1}{8}W + \tfrac{1}{3}W'\right] = \dfrac{L}{nA}\left[\tfrac{1}{2}W + w'\right]$

$\dfrac{L^3}{K^2}\dfrac{1}{8}W = \dfrac{LW}{2}$

$\dfrac{L^2}{8K^2} = \dfrac{1}{2}$

$L^2 = 4k^2$

$L^2 = 4\dfrac{l^2}{12}$

$= \dfrac{l^2}{3}$

$L = \dfrac{l}{\sqrt{3}}$

$\boxed{L = \dfrac{l\sqrt{3}}{3}}$

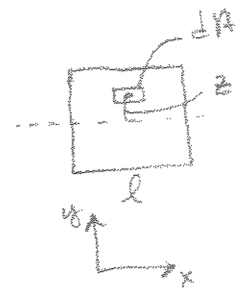
$K^2 = \dfrac{1}{A}\iint z^2\, dA$

$= \dfrac{1}{l^2}\int_{-l/2}^{l/2}\int_{-l/2}^{l/2} y^2\, dx\, dy$

$= \dfrac{1}{l}\,\dfrac{y^3}{3}\Big|_{-l/2}^{l/2}$

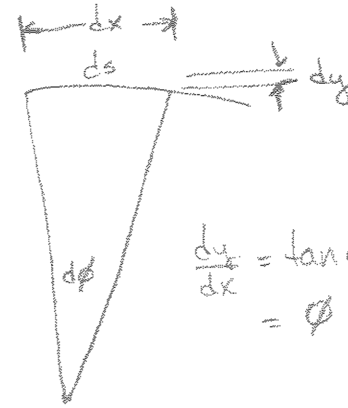
$= \dfrac{1}{l}\left[\dfrac{l^3}{24} + \dfrac{l^3}{24}\right]$

$= \dfrac{l^2}{12}$



← horizontal axis through centroid

b)



$$\frac{dy}{dx} = \tan\phi$$
$$= \phi \quad \{\phi \text{ small}\}$$

$$\frac{d\phi}{ds} = \frac{d\phi}{dx}$$
$$= \frac{d}{dx}\phi$$
$$= \frac{d}{dx}\frac{dy}{dx}$$
$$= \frac{d^2y}{dx^2}$$

$$\frac{d\phi}{dx} = \frac{d^2y}{dx^2}$$

$$d\phi = \frac{d^2y}{dx^2}dx$$

$$\phi = \int\left(\frac{d^2y}{dx^2}\right)dx \qquad y = \frac{-L^3}{YK^2A}\left[\frac{Wx^2}{4L^2} - \frac{2Wx^3}{12L^3} + \frac{Wx^4}{24L^4}\right] - \frac{L}{nA}\left[\frac{Wx}{L} - \frac{Wx^2}{2L^2}\right] \quad (5.16A)$$

$$\frac{dy}{dx} = \frac{-L^3}{YK^2A}\left[\frac{Wx}{2L^2} - \frac{Wx^2}{2L^3} + \frac{Wx^3}{6L^4}\right] - \frac{L}{nA}\left[\frac{W}{L} - \frac{Wx}{L^2}\right]$$

$$\frac{d^2y}{dx^2} = \frac{-L^3}{YK^2A}\left[\frac{W}{2L^2} - \frac{Wx}{L^3} + \frac{Wx^2}{2L^4}\right] - \frac{L}{nA}\left[\frac{W}{L^2}\right]$$

$$\phi = \int_0^L\left[\frac{-L^3}{YK^2A}\left[\frac{W}{2L^2} - \frac{Wx}{L^3} + \frac{Wx^2}{2L^4}\right] - \frac{L}{nA}\left[\frac{W}{L^2}\right]\right]dx$$

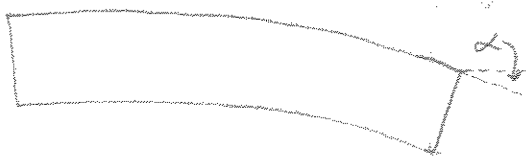
$$= \frac{-L^3}{YK^2A}\left[\frac{Wx}{2L^2} - \frac{Wx^2}{2L^3} + \frac{Wx^3}{6L^4}\right]\Big|_0^L - \frac{L}{nA}\frac{Wx}{L^2}\Big|_0^L$$

$$= \frac{-L^3}{YK^2A}\left[\frac{W}{2L} - \frac{W}{2L} + \frac{W}{6L}\right] - \frac{W}{nA}$$

$$= \frac{-L^3}{YK^2A}\left[\frac{W}{6L}\right] - \frac{W}{nA} \qquad K = \frac{\ell^2}{12} \quad A = \ell^2$$

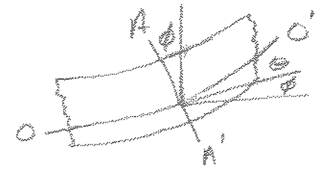
$$\boxed{\phi = \frac{-2WL^2}{Y\ell^4}}$$

$\frac{W}{nA}$ is shear and does not contribute to face angle c)



$\alpha = \theta + \phi$

$\frac{dy}{dx} = \alpha$



$$y = -\frac{L^3}{YK^2A}\left[\frac{Wx^2}{4L^2}\left(1 - \frac{2x}{3L} + \frac{1}{6}\frac{x^2}{L^2}\right)\right] - \frac{L}{nA}\left[\frac{Wx}{L}\left(1 - \frac{1}{2}\frac{x}{L}\right)\right] \{W'=0\} \quad (5.164)$$

$$y = -\frac{L^3}{YK^2A}\left[\frac{Wx^2}{4L^2} - \frac{2Wx^3}{12L^3} + \frac{Wx^4}{24L^4}\right] - \frac{L}{nA}\left[\frac{Wx}{L} - \frac{Wx^2}{2L^2}\right]$$

$$\frac{dy}{dx} = \frac{-L^3}{YK^2A}\left[\frac{Wx}{2L^2} - \frac{Wx^2}{2L^3} + \frac{Wx^3}{6L^4}\right] - \frac{L}{nA}\left[\frac{W}{L} - \frac{Wx}{L^2}\right]$$

$$\frac{dy}{dx}_{x=L} = \frac{-L^3}{YK^2A}\left[\frac{W}{2L} - \frac{W}{2L} + \frac{W}{6L}\right] - \frac{L}{nA}\left[\frac{W}{L} - \frac{W}{L}\right]$$

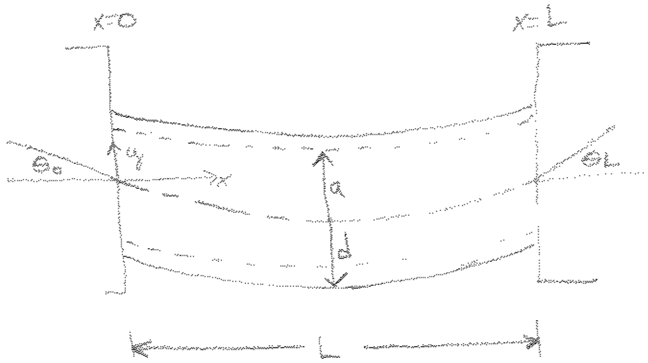
$$\alpha = \frac{-L^3}{YK^2A}\left[\frac{W}{6L}\right]$$

$$= \frac{-L^2 W}{6YK^2A} \qquad K^2 = \frac{l^2}{12}, \quad A = l^2$$

$$= \frac{-L^2 W}{6Y\frac{l^4}{12}}$$

$$\boxed{\alpha = \frac{-2L^2W}{Yl^4}}$$

5-39



$$y'''' = \frac{1}{nA}\frac{d^2w}{dx^2} - \frac{w}{YK^2A}$$

$$y''' = \frac{1}{nA}\frac{dw}{dx} - \frac{wx}{YK^2A} + C_3$$

$$y'' = \frac{1}{nA}w - \frac{w}{YK^2A}\frac{x^2}{2} + C_3x + C_2$$

$$y' = \frac{w}{nA}x - \frac{w}{YK^2A}\frac{x^3}{6} + C_3\frac{x^2}{2} + C_2x + C_1$$

$$y = \frac{w}{nA}\frac{x^2}{2} - \frac{w}{YK^2A}\frac{x^4}{24} + C_3\frac{x^3}{6} + C_2\frac{x^2}{2} + C_1x + C_0$$

B.C.

$y(0)=0,\ y(L)=0,\ y'(\tfrac{L}{2})=0,\ -y'(0)=y'(L)$

$y(0)=0:\quad 0 = C_0$

$y(L)=0:\quad 0 = \frac{w}{nA}\frac{L^2}{2} - \frac{w}{24YK^2A}L^4 + C_3\frac{L^3}{6} + C_2\frac{L^2}{2} + C_1L$

$$C_3\frac{L^3}{6} + C_2\frac{L^2}{2} + C_1L = -\frac{WL}{2nA} + \frac{WL^3}{24YK^2A} \quad \{w = \tfrac{W}{L}\}$$

$y'(\tfrac{L}{2})=0:\quad 0 = \frac{w}{nA}\frac{L}{2} - \frac{w}{6YK^2A}\frac{L^3}{8} + C_3\frac{L^2}{8} + C_2\frac{L}{2} + C_1$

$$C_3\frac{L^2}{8} + C_2\frac{L}{2} + C_1 = -\frac{W}{2nA} + \frac{WL^2}{48YK^2A}$$

$-y'(0)=y'(L):\quad -C_1 = \frac{w}{nA}L - \frac{w}{6YK^2A}L^3 + C_3\frac{L^2}{2} + C_2L + C_1$

$$C_3\frac{L^2}{2} + C_2L + 2C_1 = -\frac{W}{nA} + \frac{WL^2}{6YK^2A}$$

① $C_3\frac{L^2}{6} + C_2\frac{L}{2} + C_1 = \frac{WL^2}{24YK^2A} - \frac{W}{2nA}$

② $C_3\frac{L^2}{8} + C_2\frac{L}{2} + C_1 = \frac{WL^2}{48YK^2A} - \frac{W}{2nA}$

③ $C_3\frac{L^2}{2} + C_2 L + 2C_1 = \frac{WL^2}{6YK^2A} - \frac{W}{nA}$

① $C_3\frac{L^2}{6} + C_2\frac{L}{2} + C_1 = \frac{WL^2}{24YK^2A} - \frac{W}{2nA}$

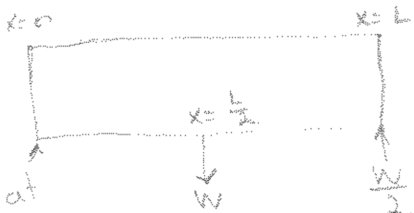
② $-C_3\frac{L^2}{8} - C_2\frac{L}{2} - C_1 = \frac{-WL^2}{48YK^2A} + \frac{W}{2nA}$

$C_3\frac{L^2}{24} = \frac{WL^2}{48YK^2A}$

$$\boxed{C_3 = \frac{W}{2YK^2A}}$$

C_2 and C_1 are indeterminant with the given boundary conditions

At $x=0$ the shear force is



pipe cut at $x=0$

$\Sigma F = 0: \; S - W + \frac{W}{2} = 0$

$S - \frac{W}{2} = 0$

$S = \frac{W}{2}$ (positive shear)

$$\boxed{y'(0) = C_1 = \Theta = \frac{S}{nA} = \frac{W}{2nA}}$$

① $C_3\frac{L^2}{6} + C_2\frac{L}{2} + C_1 = \frac{WL^2}{24YK^2A} - \frac{W}{2nA}$

$\frac{W}{2YK^2A}\frac{L^2}{6} + C_2\frac{L}{2} + \frac{W}{2nA} = \frac{WL^2}{24YK^2A} - \frac{W}{2nA}$

$$\frac{WL^2}{12YK^2A} + C_2\frac{L}{2} + \frac{W}{2nA} = \frac{WL^2}{24YK^2A} - \frac{W}{2nA}$$

$$C_2 = \frac{-WL}{6YK^2A} + \frac{WL}{12YK^2A} - \frac{2W}{nAL}$$

$$= \frac{-WL}{12YK^2A} - \frac{2W}{nAL}$$

$$y = \frac{W}{2nAL}x^2 - \frac{W}{24YK^2AL}x^4 + \frac{W}{12YK^2A}x^3 - \left(\frac{WL}{12YK^2A} + \frac{2W}{nAL}\right)\frac{x^2}{2} + \frac{W}{2nA}x$$

$$y\left(\frac{L}{2}\right) = \frac{W}{2nAL}\frac{L^2}{4} - \frac{W}{24YK^2AL}\frac{L^4}{16} + \frac{W}{12YK^2A}\frac{L^3}{8} - \frac{WL}{24YK^2A}\frac{L^2}{4} - \frac{2W}{nAL}\frac{L^2}{4} + \frac{W}{2nA}\frac{L}{2}$$

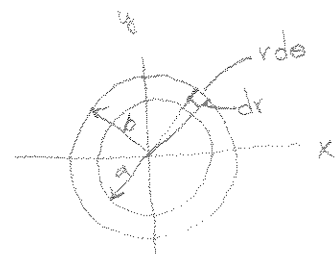
$$= \frac{-WL^3}{384YK^2A} + \frac{WL^3}{96YK^2A} - \frac{WL^3}{96YK^2A} + \frac{WL}{8nA} - \frac{WL}{2nA} + \frac{WL}{4nA}$$

$$= \frac{-WL^3}{384YK^2A} + \frac{4WL^3}{384YK^2A} - \frac{4WL^3}{384YK^2A} - \frac{WL}{8nA}$$

$$= \frac{-WL^3}{384YK^2A} - \frac{WL}{8nA}$$

$$= \frac{-\rho L^4\pi(b^2-a^2)}{384Y\frac{\pi}{4}(b^4-a^4)} - \frac{\rho L^2}{8n}$$

$$\boxed{y\left(\frac{L}{2}\right) = \frac{-\rho L^4}{96Y(b^2+a^2)} - \frac{\rho L^2}{8n}}$$



$$K_z^2 = \frac{1}{A}\int r^2\,dA$$

$$= \frac{1}{A}\int_0^{2\pi}\int_a^b r^3\,dr\,d\theta$$

$$= \frac{2\pi}{A}\frac{r^4}{4}\Big|_a^b$$

$$= \frac{\pi}{2A}(b^4-a^4)$$

$I_x + I_y = I_z$ {⊥ axis theorem}

$$I_x = I_y$$

$$2I_x = I_z$$

$$2K_x = K_z$$

$$K_x = \frac{K_z}{2}$$

$$= \frac{\pi}{4A}(b^4-a^4)$$

$$W = \rho AL$$

$$W = \rho\pi(b^2-a^2)L$$

Find increase in deflection when the pipe is filled with a fluid of density ρ_0

Since a fluid does not support shear forces it only adds to the distributed load w.

$$y\left(\tfrac{L}{2}\right) = \frac{-WL^3}{384YK^2A} - \frac{WL}{8nA}$$

$$= \frac{-WL^3}{96Y\pi(b^4-a^4)} - \frac{WL}{8n\pi(b^2-a^2)}$$

$$\Delta y\left(\tfrac{L}{2}\right) = \frac{-\Delta WL^2}{96Y\pi(b^4-a^4)} - \frac{\Delta WL}{8n\pi(b^2-a^2)}$$

ΔW = addition of fluid

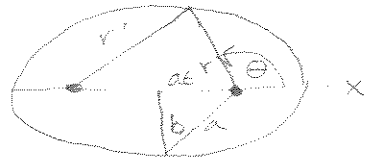
$= \rho_0 \pi a^2 L$

$$\Delta y\left(\tfrac{L}{2}\right) = \frac{-\rho_0 \pi a^2 L^3}{96Y\pi(b^4-a^4)} - \frac{\rho_0 \pi a^2 L^2}{8n\pi(b^2-a^2)}$$

$$\boxed{\Delta y\left(\tfrac{L}{2}\right) = \frac{-\rho_0 a^2 L^3}{96Y(b^4-a^4)} - \frac{\rho_0 a^2 L^2}{8n(b^2-a^2)}}$$

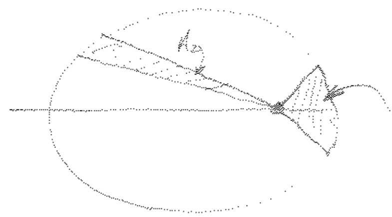
6-1

a) Keplar's 1^{st} Law: $r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\theta}$



Keplar's 2^{nd} Law:

$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$



$A_1 = A_2$ for $t_1 = t_2$

$dA = \frac{1}{2}r^2 d\theta$

$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta}$

Newton's Laws: $\vec{F} = m\vec{a}$, $\vec{F} = -\vec{F}$

$\vec{r} = r\hat{r}(\theta)$

$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \dot{r}\hat{r} + r\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt}$ $\quad \frac{d\hat{r}}{d\theta} = \hat{\theta}$, $\frac{d\hat{\theta}}{d\theta} = -\hat{r}$

$\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$

$\ddot{\vec{r}} = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} + \dot{r}\dot{\theta}\hat{\theta} + r\left[\ddot{\theta}\hat{\theta} + \dot{\theta}\frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt}\right]$

$= \ddot{r}\hat{r} + \dot{r}\dot{\theta}\hat{\theta} + \dot{r}\dot{\theta}\hat{\theta} + r\ddot{\theta}\hat{\theta} - r\dot{\theta}^2\hat{r}$

$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\hat{\theta}$

$r = p(1+\epsilon\cos\theta)^{-1}$ $\quad \{p = a(1-\epsilon^2)\}$

Keplar's 2^{nd} Law

$\frac{d}{dt}\left(\frac{1}{2}r^2\dot{\theta}\right) = 0$ $\quad \{\frac{dA}{dt} = \text{constant}\}$

$\frac{1}{2}r^2\dot{\theta} = C$

$\dot{\theta} = \frac{2C}{r^2} = \frac{K}{r^2} \Rightarrow \ddot{\theta} = \frac{-2K}{r^3}\dot{r}$

$$\ddot{\vec{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (r\ddot{\theta} + 2\dot{r}\ddot{\theta})\hat{\theta}$$

$$= \left(\ddot{r} - r\left(\frac{K}{r^2}\right)^2\right)\hat{r} + \left(r\left(\frac{-2k}{r^3}\dot{r}\right) + 2\dot{r}\left(\frac{K}{r^2}\right)\right)\hat{\theta}$$

$$= \left(\ddot{r} - \frac{K^2}{r^3}\right)\hat{r} + \left(-\frac{2K}{r^2}\dot{r} + \frac{2K}{r^2}\dot{r}\right)\hat{\theta}$$

$$= \left(\ddot{r} - \frac{K^2}{r^3}\right)\hat{r}$$

$$r = p(1+\epsilon\cos\theta)^{-1}$$

$$\dot{r} = p\epsilon\sin\theta(1+\epsilon\cos\theta)^{-2}\dot{\theta}$$

$$= p\epsilon\sin\theta(1+\epsilon\cos\theta)^{-2}\frac{K}{r^2}$$

$$= \frac{p}{1+\epsilon\cos\theta}\,\frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\,\frac{K}{r^2}$$

$$= \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\,\frac{K}{r}$$

$$\ddot{r} = \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\left(\frac{-K}{r^2}\right)\dot{r} + \frac{K}{r}\left[\frac{(1+\epsilon\cos\theta)\epsilon\cos\theta\,\dot{\theta} + \epsilon\sin\theta(\epsilon\sin\theta)\dot{\theta}}{(1+\epsilon\cos\theta)^2}\right]$$

$$= \frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\left(\frac{-K}{r^2}\right)\frac{\epsilon\sin\theta}{1+\epsilon\cos\theta}\left(\frac{K}{r}\right) + \frac{K}{r}\,\frac{K}{r^2}\left[\frac{\epsilon\cos\theta}{(1+\epsilon\cos\theta)} + \frac{(\epsilon\sin\theta)^2}{(1+\epsilon\cos\theta)^2}\right]$$

$$= -\frac{K^2}{r^3}\,\frac{(\epsilon\sin\theta)^2}{(1+\cos\theta)^2} + \frac{K^2}{r^3}\,\frac{\epsilon\cos\theta}{(1+\epsilon\cos\theta)} + \frac{K^2}{r^3}\,\frac{(\epsilon\sin\theta)^2}{(1+\epsilon\cos\theta)^2}$$

$$= \frac{K^2}{r^3}\,\frac{\epsilon\cos\theta}{(1+\epsilon\cos\theta)}$$

$$\ddot{\vec{r}} = \left[\frac{K^2}{r^3}\left(\frac{\epsilon\cos\theta}{1+\epsilon\cos\theta} - 1\right)\right]\hat{r} \qquad \frac{r}{p} = (1+\epsilon\cos\theta)^{-1}$$

$$\frac{p}{r} = 1+\epsilon\cos\theta$$

$$= \frac{K^2}{r^3}\left(\frac{\frac{p-r}{r}}{1+\frac{p-r}{r}} - 1\right)\hat{r} \qquad \frac{p}{r} - 1 = \epsilon\cos\theta$$

$$\frac{K^2}{r^3}\left(\frac{\frac{p-r}{r}}{\frac{p}{r}} - 1\right)\hat{r} \qquad \frac{p-r}{r} = \epsilon\cos\theta$$

$$= \frac{K^2}{r^3}\left(\frac{r}{p}\left(\frac{p-r}{r}\right) - 1\right)\hat{r}$$

$$= \frac{K^2}{r^3}\left(\frac{p-r}{p} - 1\right)\hat{r}$$

$$= \frac{K^2}{r^3}\left(\frac{p-r-p}{p}\right)\hat{r}$$

$$\ddot{\vec{r}} = \frac{K^2}{r^3}\left(\frac{-r}{p}\right)\hat{r}$$

$$= \frac{-K^2}{p}\frac{1}{r^2}\hat{r}$$

$$m\ddot{\vec{r}} = \frac{-K^2}{p}\frac{m}{r^2}\hat{r}$$

$$\boxed{m\ddot{\vec{r}} = \frac{-K_0 m}{r^2}\hat{r}, \; K_0 = \frac{K^2}{p}}$$

b) Kepler's 3^{rd} Law:

$$\tau^2 = Ka^3$$

$$L^2 = \frac{4\pi^2 a^2 b^2 m^2}{\tau^2}$$

$$L^2 = \frac{4\pi^2 b^2 m^2}{Ka}$$

$$m\ddot{r} - mr\dot{\theta}^2 = F(r)$$

$$m\ddot{r} - mr\frac{L^2}{m^2r^4} = F(r)$$

$$m\ddot{r} - \frac{L^2}{mr^3} = F(r)$$

$$m\ddot{r} - \frac{4\pi^2 b^2 m^2}{Ka}\frac{1}{mr^3} = F(r)$$

$$m\ddot{r} - \frac{mC}{r^3} = F(r)$$

$$\therefore F(r) \propto m$$

$$\frac{dS}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{L}{2m} \quad \left\{\dot{\theta} = \frac{L}{mr^2}\right\}$$

$$S = \frac{L\tau}{2m}$$

$$\pi ab = \frac{L\tau}{2m}$$

$$L^2 = \frac{\pi^2 a^2 b^2 4m^2}{\tau^2}$$

c) $F_1 = F_2$

$$F_1 = \frac{m_1 K_1}{r^2} \qquad F_2 = \frac{m_2 K_2}{r^2}$$

$$m_1 K_1 = m_2 K_2$$

$$\frac{K_1}{m_2} = \frac{K_2}{m_1} = K_3$$

$$K_1 = K_3 m_2$$

$$K_2 = K_3 m_1$$

$F_1 = F_2$

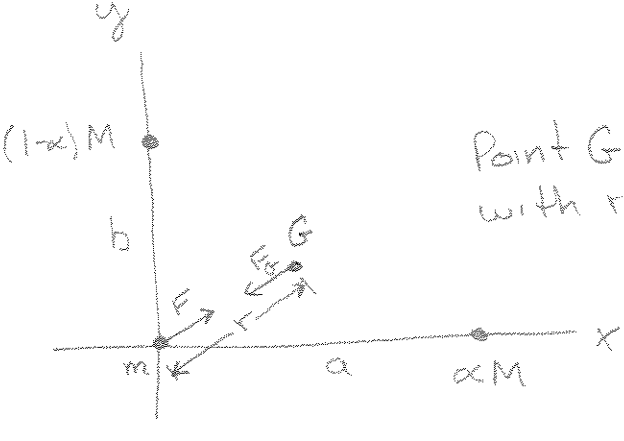
$$F_1^2 = F_1 \cdot F_2$$

$$= \frac{m_1 m_2 K_3}{r^2} \cdot \frac{m_2 m_1 K_3}{r^2}$$

$$= \frac{m_1^2 m_2^2 K_3^2}{r^4}$$

$$\boxed{\begin{aligned} F_1 &= K_3 \frac{m_1 m_2}{r^2} \\ F_2 &= K_3 \frac{m_1 m_2}{r^2} \end{aligned}}$$

3. A mass αM is located at $x = a$, $y = 0$, and a second mass $(1-\alpha)M$ is located at $x = 0$, $y = b$, where $0 < \alpha < 1$. Find the coordinates x, y of the center of gravity of the two masses relative to the origin. Show that your formulas for x, y have the proper limits when $\alpha \to 0$ or $b \to \infty$.



Point G is center of gravity with respect to origin.

$$\vec{F} = \sum_i \frac{m m_i G(\vec{r}_i - \vec{r}_0)}{|\vec{r}_i - \vec{r}_0|^3} \qquad \vec{r}_0 = 0 \text{ (origin)}$$

$$\vec{F} = \frac{m\alpha M G a\hat{x}}{a^3} + \frac{m(1-\alpha)MGb\hat{y}}{b^3}$$

$$F = \left[\left(\frac{m\alpha MGa}{a^3}\right)^2 + \left(\frac{m(1-\alpha)MGb}{b^3}\right)^2\right]^{1/2}$$

$$= \left[\frac{m^2\alpha^2M^2G^2}{a^4} + \frac{m^2(1-\alpha)^2M^2G^2}{b^4}\right]^{1/2}$$

$$F = mMG\left[\frac{\alpha^2}{a^4} + \frac{(1-\alpha)^2}{b^4}\right]^{1/2}$$

$$F_G = \frac{mMG}{r^2}$$

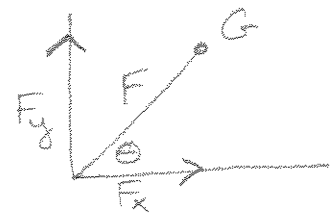
$F_G = F$ (Newton's Third Law)

$$\frac{mMG}{r^2} = mMG\left[\frac{b^4\alpha^2 + a^4(1-\alpha)^2}{a^4b^4}\right]^{1/2}$$

$$\frac{1}{r^2} = \left[\frac{b^4\alpha^2 + a^4(1-\alpha)^2}{a^4 b^4}\right]^{1/2}$$

$$r^2 = \left[\frac{a^4 b^4}{b^4\alpha^2 + a^4(1-\alpha)^2}\right]^{1/2} = \frac{a^2 b^2}{\left[b^4\alpha^2 + a^4(1-\alpha)^2\right]^{1/2}}$$

$$r = \frac{ab}{\left[b^4\alpha^2 + a^4(1-\alpha)^2\right]^{1/4}}$$



$$\cos\theta = \frac{F_x}{F}$$

$$= \frac{\dfrac{mMG\alpha}{a^2}}{\dfrac{mMG\left[b^4\alpha^2 + a^4(1-\alpha)^2\right]^{1/2}}{a^2 b^2}}$$

$$\cos\theta = \frac{\alpha b^2}{\left[b^4\alpha^2 + a^4(1-\alpha^2)^2\right]^{1/2}}$$

$$\sin\theta = \frac{F_y}{F}$$

$$= \frac{\dfrac{mMG(1-\alpha)}{b^2}}{\dfrac{mMG\left[b^4\alpha^2 + a^4(1-\alpha)^2\right]^{1/2}}{a^2 b^2}}$$

$$\sin\theta = \frac{(1-\alpha)a^2}{\left[b^4\alpha^2 + a^4(1-\alpha)^2\right]^{1/2}}$$

$$x_G = r\cos\theta$$

$$= \frac{ab}{\left[b^4\alpha^2 + a^4(1-\alpha)^2\right]^{1/4}}\cos\theta$$

$$= \frac{ab^3\alpha}{\left[b^4\alpha^2 + a^4(1-\alpha)^2\right]^{3/4}}$$

$$\boxed{x_G = \frac{a\alpha}{\left[\alpha^2 + \frac{a^4}{b^4}(1-\alpha)^2\right]^{3/4}}}$$

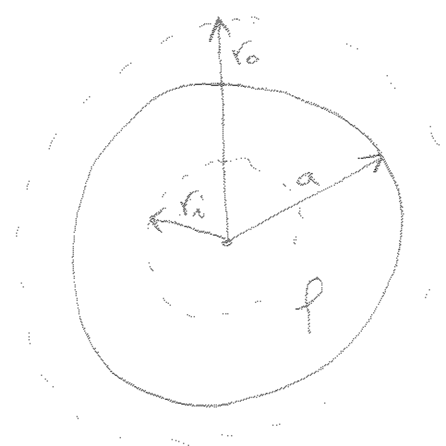
$$y_G = r\sin\theta$$

$$= \frac{ab}{\left[b^4\alpha^2 + a^4(1-\alpha)^2\right]}\sin\theta$$

$$= \frac{a^3 b(1-\alpha)}{\left[b^4\alpha^2 + a^4(1-\alpha)^2\right]^{3/4}}$$

$$\boxed{y_G = \frac{b(1-\alpha)}{\left[\frac{b^4}{a^4}\alpha^2 + (1-\alpha)^2\right]^{3/4}}}$$

5. Assuming that the earth is a sphere of uniform density, with radius a, mass M, calculate the gravitational field intensity and the gravitational potential at all points inside and outside the earth, taking $\mathcal{G} = 0$ at an infinite distance.



a) Calculate $\vec{g}$

$$\int \vec{g}\cdot d\vec{S} = -4\pi GM$$

$r \geq a$ (outside)

$$\int \vec{g}\cdot d\vec{S} = -4\pi GM$$

$$g\,4\pi r^2 = -4\pi GM$$

$$g = -\frac{MG}{r^2}$$

$$\boxed{\vec{g} = -\frac{MG}{r^2}\frac{\vec{r}}{r}}$$

$r \leq a$ (inside)

$$m = \rho \tfrac{4}{3}\pi r^3$$

$$\int \vec{g}\cdot d\vec{S} = -4\pi Gm$$

$$g\,4\pi r^2 = -4\pi G\rho \tfrac{4}{3}\pi r^3$$

$$g = -G\rho\tfrac{4}{3}\pi r \qquad M = \rho\tfrac{4}{3}\pi a^3$$

$$\frac{M}{a^3} = \rho\tfrac{4}{3}\pi$$

$$\vec{g} = -\frac{MGr}{a^3}\frac{\vec{r}}{r}$$

$$\boxed{\vec{g} = -\frac{MG\vec{r}}{a^3}}$$

b) Calculate U

$$U = \int_{r_s}^{r} \vec{g}\cdot d\vec{r}$$

$$U_o = \int_{\infty}^{r} -\frac{MG}{r'^2}\,dr'$$

$$= -MG\int_{\infty}^{r} r'^{-2}\,dr'$$

$$= MG\frac{1}{r'}\Big|_{\infty}^{r}$$

$$\boxed{U_o = \frac{MG}{r} \quad (r \geq a)}$$

$$U_i = \int_{r_s}^{r} -\frac{MGr'}{a^3}\,dr'$$

$$= -\frac{MG}{a^3}\int_{r_s}^{r} r'\,dr'$$

$$= -\frac{MG}{a^3}\,\frac{r'^2}{2}\Big|_{r_s}^{r}$$

$$= -\frac{MG}{a^3}\left[\frac{r^2}{2} - \frac{r_s^2}{2}\right]$$

$$= \frac{MG}{2a^3}\left[r_s^2 - r^2\right]$$

To determine r_s, $U_o = U_i$ at $r = a$

$$U_{o\,r=a} = \frac{MG}{a} \qquad U_{i\,r=a} = \frac{MG}{2a^3}\left[r_s^2 - a^2\right]$$

$$\frac{MG}{a} = \frac{MG}{2a^3}\left[r_s^2 - a^2\right]$$

$$2a^2 = r_s^2 - a^2$$

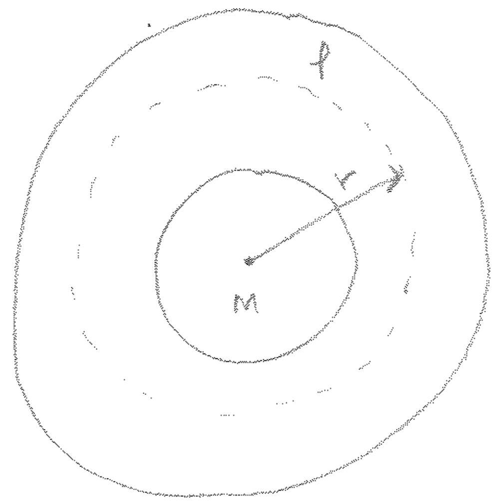
$$3a^2 = r_s^2$$

$$\boxed{U_i = \frac{MG}{2a^3}\left[3a^2 - r^2\right]}$$

7. Show that if the sun were surrounded by a spherical cloud of dust of uniform density ρ, the gravitational field within the dust cloud would be

$$\boldsymbol{g} = -\left(\frac{MG}{r^2} + \frac{4\pi}{3}\rho G r\right)\frac{\boldsymbol{r}}{r},$$

where M is the mass of the sun, and $\boldsymbol{r}$ is a vector from the sun to any point in the dust cloud.



Outside the Sun

$$\oint \vec{g}\cdot d\vec{s} = -4\pi G M$$

$$g\,4\pi r^2 = -4\pi G M$$

$$g = \frac{-MG}{r^2}$$

$$\vec{g}_s = -\frac{MG}{r^2}\frac{\vec{r}}{r}$$

Inside dust cloud

$$\oint \vec{g}\cdot d\vec{s} = -4\pi G m \qquad m = \rho\frac{4}{3}\pi r^3$$

$$g\,4\pi r^2 = -4\pi G\rho\frac{4}{3}\pi r^3$$

$$g = -\frac{4}{3}\rho\pi G r$$

$$\vec{g}_d = -\frac{4\pi}{3}\rho G r\frac{\vec{r}}{r}$$

$$\vec{g} = \vec{g}_s + \vec{g}_d$$

$$\boxed{\vec{g} = -\left(\frac{MG}{r^2} + \frac{4\pi}{3}\rho G r\right)\frac{\vec{r}}{r}}$$

6-9

$$\vec{\nabla}P = \rho g \qquad\qquad g = -\frac{GM}{r^2} \ \{\text{spherical body}\}$$

$$\frac{dP}{dr}\frac{1}{\rho} = g \qquad\qquad \frac{dg}{dr} = \frac{-G}{r^2}\frac{dM}{dr}\ \frac{2GM}{r}$$

$$\frac{dg}{dr} = \frac{d^2P}{dr^2}\frac{1}{\rho} + \frac{dP}{dr}\frac{d}{dr}\left(\frac{1}{\rho}\right)$$

$$\frac{d^2P}{dr^2}\frac{1}{\rho} + \frac{dP}{dr}\frac{d}{dr}\left(\frac{1}{\rho}\right) = \frac{-G\,dM}{r^2\,dr} + \frac{2GM}{r}$$

$$\frac{1}{\rho}\frac{d^2P}{dr^2} + \frac{dP}{dr}\frac{d}{dr}\left(\frac{1}{\rho}\right) - \frac{2GM}{r} = -\frac{G\,dM}{r^2\,dr}$$

$$\frac{1}{\rho}\frac{d^2P}{dr^2} + \frac{dP}{dr}\frac{d}{dr}\left(\frac{1}{\rho}\right) + GM\frac{d}{dr}\left(\frac{1}{r^2}\right) = -\frac{G\,dM}{r^2\,dr}$$

$$\frac{r^2}{\rho}\frac{d^2P}{dr^2} + \frac{dP}{dr}\frac{d}{dr}\left(\frac{r^2}{\rho}\right) + GM\frac{d}{dr}\left(\frac{r^2}{r^2}\right) = -G\frac{dM}{dr}$$

$$\frac{r^2}{\rho}\frac{d^2P}{dr^2} + \frac{dP}{dr}\frac{d}{dr}\left(\frac{r^2}{\rho}\right) = -G\frac{dM}{dr} \qquad\qquad \frac{dM}{dr} = \rho\, dV = \rho\, 4\pi r^2$$

$$\frac{r^2}{\rho}\frac{d^2P}{dr^2} + \frac{dP}{dr}\frac{d}{dr}\left(\frac{r^2}{\rho}\right) = -4\pi r^2 G\rho \qquad\qquad \rho = \frac{AP}{RT}$$

$$\frac{RTr^2}{AP}\frac{d^2P}{dr^2} + \frac{dP}{dr}\frac{d}{dr}\left(\frac{RTr^2}{AP}\right) = -4\pi r^2 G\frac{AP}{RT} \qquad\qquad \frac{1}{\rho} = \frac{RT}{AP}$$

$$\frac{R^2}{A^2}\frac{Tr^2}{P}\frac{d^2P}{dr^2} + \frac{R^2}{A^2}\frac{dP}{dr}\frac{d}{dr}\left(\frac{Tr^2}{P}\right) = -4\pi r^2 G\frac{P}{T}$$

$$\frac{d}{dr}\left(\frac{R^2T}{A^2P}r^2\frac{dP}{dr}\right) = -4\pi r^2 G\frac{P}{T}$$

$$\boxed{T\frac{d}{dr}\left(\frac{R^2Tr^2}{A^2P}\frac{dP}{dr}\right) = -4\pi r^2 GP}$$

B.C.: $M = \int_0^\infty \rho\, dV$

$$M = \int_0^\infty \frac{AP}{RT}4\pi r^2 dr$$

$\rho(\infty) = 0 \qquad\qquad A = \frac{M}{n}$

6-11

$$\rho = \frac{Ma^2}{2\pi r(r^2+a^2)^2},\ 0 \le r < \infty$$

$$\vec{f} = \rho \vec{g}$$

$$\vec{\nabla} P = \vec{f}$$

$$\vec{\nabla} P = -\rho g$$

$$\frac{dP}{dr} = -\rho g$$

For a spherical mass

$$g = \frac{GM(r)}{r^2} \qquad M(r) = \int_0^r \rho\, dV$$

$$= \int_0^r \frac{Ma^2}{2\pi r'(r'^2+a^2)^2} 4\pi r'^2 dr'$$

$$= Ma^2 \int_0^r \frac{2r'dr'}{(r'^2+a^2)^2} \quad \text{let } u = r'^2+a^2, \ du = 2r'dr'$$

$$= Ma^2 \int \frac{du}{u^2}$$

$$= Ma^2 \frac{-1}{u}$$

$$M(r) = -Ma^2 \frac{1}{r^2+a^2}\Big|_0^r$$

$$M(r) = -Ma^2\left[\frac{1}{r^2+a^2} - \frac{1}{a^2}\right]$$

$$\frac{dP}{dr} = \frac{-Ma^2}{2\pi r(r^2+a^2)^2} \frac{-GMa^2}{r^2}\left[\frac{1}{r^2+a^2} - \frac{1}{a^2}\right]$$

$$dP = \frac{GM^2a^4}{2\pi}\left[\frac{1}{r^3(r^2+a^2)^3} - \frac{1}{a^2r^3(r^2+a^2)^2}\right]dr$$

$$P = \frac{GM^2a^4}{2\pi}\left[\int \frac{dr}{r^3(r^2+a^2)^3} - \frac{1}{a^2}\int \frac{dr}{r^3(r^2+a^2)^2}\right]$$

Evaluate $\int \frac{dr}{r^3(r^2+a^2)^3}$

$$\int \frac{dr}{r^3(r^2+a^2)^3} = \int \frac{a\sec^2\theta d\theta}{a^3\tan^3\theta(a^2\tan^2\theta+a^2)^3}$$

let $r = a\tan\theta$, $dr = a\sec^2\theta d\theta$

$$= \int \frac{\sec^2\theta d\theta}{a^2\tan^3\theta(a^2\sec^2\theta)^3}$$

$$= \frac{1}{a^8}\int \frac{d\theta}{\tan^3\theta \sec^4\theta} = \frac{1}{a^8}\int \frac{\cos^7\theta d\theta}{\sin^3\theta}$$

$$= \frac{1}{a^8}\int \frac{\cos^5\theta(1-\sin^2\theta)d\theta}{\sin^3\theta}$$

$$= \frac{1}{a^8}\left[\int \frac{\cos^5\theta d\theta}{\sin^3\theta} - \int \frac{\cos^5\theta}{\sin\theta}d\theta\right]$$

Evaluate $\frac{1}{a^2}\int \frac{dr}{r^3(r^2+a^2)^2}$

$$\frac{1}{a^2}\int \frac{dr}{r^3(r^2+a^2)^2} = \frac{1}{a^2}\int \frac{a\sec^2\theta d\theta}{a^3\tan^3\theta(a^2\tan^2\theta+a^2)^2}$$

let $r = a\tan\theta$, $dr = a\sec^2\theta d\theta$

$$= \frac{1}{a^2}\int \frac{\sec^2\theta d\theta}{a^2\tan^3\theta(a^2\sec^2\theta)^2}$$

$$= \frac{1}{a^8}\int \frac{d\theta}{\tan^3\theta \sec^2\theta} = \frac{1}{a^8}\int \frac{\cos^5\theta d\theta}{\sin^3\theta}$$

$$\int \frac{dr}{r^3(r^2+a^2)^3} - \frac{1}{a^2}\int \frac{dr}{r^3(r^2+a^2)^2} = \frac{1}{a^8}\left[\int \frac{\cos^5\theta d\theta}{\sin^3\theta} - \int \frac{\cos^5\theta}{\sin\theta}d\theta - \int \frac{\cos^5\theta d\theta}{\sin^3\theta}\right]$$

$$= -\frac{1}{a^8}\int \frac{\cos^5\theta d\theta}{\sin\theta} = -\frac{1}{a^8}\int \frac{\cos^3\theta(1-\sin^2\theta)d\theta}{\sin\theta}$$

$$= -\frac{1}{a^8}\left[\int \frac{\cos^3\theta}{\sin\theta}d\theta - \int \cos^3\theta\sin\theta d\theta\right]$$

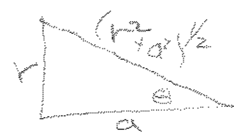
$$= -\frac{1}{a^8}\left[\int \frac{\cos\theta(1-\sin^2\theta)d\theta}{\sin\theta} - \int \cos^3\theta\sin\theta d\theta\right]$$

$$= -\frac{1}{a^8}\left[\int \frac{\cos\theta}{\sin\theta}d\theta - \int \cos\theta\sin\theta d\theta - \int \cos^3\theta\sin\theta d\theta\right]$$

$u = \sin\theta$, $du = \cos\theta d\theta$ $\quad$ $u' = \cos\theta$, $-du' = \sin\theta d\theta$ $\quad$ $u' = \cos\theta$, $-du' = \sin\theta d\theta$

$$= \frac{-1}{a^8}\left[\int \frac{du}{u} + \int u'\,du' + \int u'^3\,du'\right]$$

$$= \frac{-1}{a^8}\left[\ln\sin\theta + \frac{\cos^2\theta}{2} + \frac{\cos^4\theta}{4}\right]$$



$$= \frac{-1}{a^8}\left[\ln\frac{r}{(r^2+a^2)^{1/2}} + \frac{a^2}{2(r^2+a^2)} + \frac{a^4}{4(r^2+a^2)^4}\right]$$

$$= \frac{1}{a^8}\left[\ln\frac{(r^2+a^2)^{1/2}}{r} - \frac{a^4}{4(r^2+a^2)^2} - \frac{a^2}{2(r^2+a^2)}\right]$$

$$= \frac{1}{2a^8}\left[\ln\frac{r^2+a^2}{r^2} - \frac{a^4}{2(r^2+a^2)^2} - \frac{a^2}{r^2+a^2}\right]$$

$$P = \frac{GM^2a^4}{4\pi a^8}\left[\ln\frac{r^2+a^2}{r^2} - \frac{a^4}{2(r^2+a^2)^2} - \frac{a^2}{r^2+a^2}\right]$$

$$\boxed{P = \frac{GM^2}{4\pi a^4}\left[\ln\frac{r^2+a^2}{r^2} - \frac{a^4}{2(r^2+a^2)^2} - \frac{a^2}{r^2+a^2}\right]}$$

$$PV = nRT \qquad \frac{M}{A} = n$$

$$PV = \frac{M}{A}RT$$

$$\frac{AP}{RT} = \frac{M}{V} = \rho$$

$$T = \frac{A}{R}\frac{P}{\rho}$$

$$= \frac{A}{R}\,\frac{2\pi r(r^2+a^2)^2}{Ma^2}\,\frac{GM^2}{4\pi a^4}\left[\ln\frac{r^2+a^2}{r^2} - \frac{a^4}{2(r^2+a^2)^2} - \frac{a^2}{r^2+a^2}\right]$$

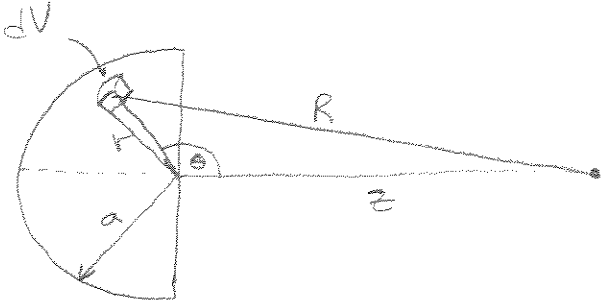
$$= \frac{AGMr}{2Ra^2}\left[\frac{(r^2+a^2)^2}{a^4}\ln\frac{r^2+a^2}{r^2} - \frac{1}{2} - \frac{r^2+a^2}{a^2}\right]$$

$$\boxed{T = \frac{AGMr}{2Ra^2}\left[\frac{(r^2+a^2)^2}{a^4}\ln\frac{r^2+a^2}{r^2} - \frac{3}{2} - \frac{r^2}{a^2}\right]}$$

6-13

Gravitational Field of Hemisphere

a)



$dm = \rho\, dV$

$= \rho r^2 \sin\theta\, d\phi\, d\theta\, dr$

$R = (r^2 + z^2 - 2rz\cos\theta)^{1/2}$

Potential at z

$$\mathcal{G}_H = G \int \frac{dm}{R}$$

$$= 2\pi \rho G \int_0^a \int_{\pi/2}^{\pi} \frac{r^2 \sin\theta\, d\theta\, dr}{(r^2+z^2-2rz\cos\theta)^{1/2}}$$

$u = r^2 + z^2 - 2rz\cos\theta$

$du = 2rz\sin\theta\, d\theta$

$\frac{r\,du}{2z} = r^2 \sin\theta\, d\theta$

$$= \frac{\pi\rho G}{z} \int_0^a \int_{u_1}^{u_2} \frac{du}{u^{1/2}}\, r\,dr$$

$$= \frac{2\pi\rho G}{z} \int_0^a (r^2+z^2-2rz\cos\theta)^{1/2} \Big|_{\pi/2}^{\pi} r\,dr$$

$$= \frac{2\pi\rho G}{z} \int_0^a \left[r + z - (r^2+z^2)^{1/2} \right] r\,dr$$

$$= \frac{2\pi\rho G}{z} \left[\frac{r^3}{3} + \frac{r^2}{2} z - \frac{1}{3}(r^2+z^2)^{3/2} \right] \Big|_0^a$$

$$\mathcal{G}_H(z) = \frac{2\pi\rho G}{z} \left[\frac{a^3}{3} + \frac{a^2}{2} z - \frac{1}{3}(a^2+z^2)^{3/2} + \frac{z^3}{3} \right]$$

$$\vec{g}_H = \vec{\nabla} \mathcal{G}_H$$

$$= \frac{\partial}{\partial z} \mathcal{G}\, \hat{z}$$

$$= \frac{\partial}{\partial z}\, 2\pi\rho G \left[\frac{a^3}{3z} + \frac{a^2}{2} - \frac{(a^2+z^2)^{3/2}}{3z} + \frac{z^2}{3} \right] \hat{z}$$

$$= 2\pi\rho G \left[\frac{-a^3}{3z^2} - \frac{3z\left(\frac{3}{2}\right)(a^2+z^2)^{1/2}(2z) - 3(a^2+z^2)^{3/2}}{9z^2} + \frac{2z}{3} \right] \hat{z}$$

$$\vec{g}_H(z) = 2\pi\rho G \left[\frac{-a^3}{3z^2} - (a^2+z^2)^{1/2} + \frac{(a^2+z^2)^{3/2}}{3z^2} + \frac{2z}{3} \right] \hat{z}$$

$$\vec{g}_H(a) = 2\pi\rho G\left[-\frac{a}{3} - \sqrt{2}a + \frac{2\sqrt{2}}{3}a + \frac{2}{3}a\right]\hat{z}$$

$$= 2\pi\rho G\left[\frac{a}{3} - \frac{\sqrt{2}a}{3}\right]\hat{z} \qquad M = \frac{\rho}{2}\,\frac{4}{3}\pi a^3$$

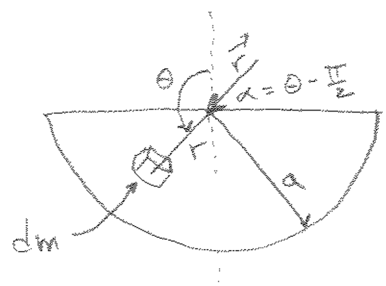
$$= \frac{MG}{a^2}\left[1 - \sqrt{2}\right]\hat{z}$$

$$g = g_0 - g_H(a) - \frac{MG}{a^2}$$

$$= g_0 - \frac{MG}{a^2}\left[2 - \sqrt{2}\right]$$

$$\boxed{g_0 - g = \frac{MG}{a^2}\left[2 - \sqrt{2}\right]}$$

b)



$$\vec{g} = \int \frac{G\rho\,\vec{r}}{|\vec{r}|^3}\,dV \qquad dm = \rho\,dV$$
$$= \rho r^2 \sin\theta\,d\phi\,d\theta\,dr$$

$$\vec{g}_y = \int \frac{G\rho\, r\sin\alpha\,\hat{y}}{r^3}\,dV \qquad \vec{r} = r(\cos\alpha\,\hat{x} + \sin\alpha\,\hat{y})$$

$$= -\int \frac{G\rho\cos\theta\,\hat{y}}{r^2}\,dV \qquad |\vec{r}|^3 = r^3$$

$$= -2\pi G\rho \int_0^a \int_{\pi/2}^{\pi} \frac{r^2 \sin\theta\cos\theta\,d\theta\,dr}{r^2}\,\hat{y} \qquad \sin\alpha = \sin\left(\theta - \frac{\pi}{2}\right) = -\cos\theta$$

$$= -2\pi G\rho \int_0^a \int_{\frac{\pi}{2}}^{\pi} \sin\theta\cos\theta\,d\theta\,dr\,\hat{y}$$

$$= -2\pi G\rho \int_0^a \left.\frac{\sin^2\theta}{2}\right|_{\pi/2}^{\pi} dr\,\hat{y}$$

$$= -\pi G\rho \int_0^a (-1)\,dr\,\hat{y}$$

$$= \pi G\rho \int_0^a dr\,\hat{y}$$

$$= \pi G\rho a\,\hat{y} \qquad M_E = \rho_e \tfrac{4}{3}\pi R_E^3 \qquad M = \tfrac{\rho_e}{2}\tfrac{4}{3}\pi a^3 \cdot \tfrac{1}{2}$$

$$g_{y_E} = \frac{3M_E G a}{4R_E^3} \qquad \frac{3M_E}{4\pi R_E^3} = \rho_e \qquad \frac{3M}{4\pi a^3} = \rho$$

$$g_{y_m} = \frac{3MGa}{4a^3}$$

$$g_{y_m} = \frac{3MG}{4a^2}$$

Field due to earth

$$g_0 = -\frac{GM_E}{R_E^2}$$

Field at center surface of hemisphere

$$g = g_0 + g_{y_E} - g_{y_m}$$

$$= g_0 + \frac{3}{4}\frac{M_e G a}{R_e^3} - \frac{3MG}{4a^2} \qquad \frac{3M_E}{4\pi R_E^3} = \frac{3M}{2\pi a^3}$$

$$= g_0 + \frac{3}{4}\frac{2MG}{a^2} - \frac{3}{4}\frac{MG}{a^2} \qquad M_e = \frac{2MR_e^3}{a^3}$$

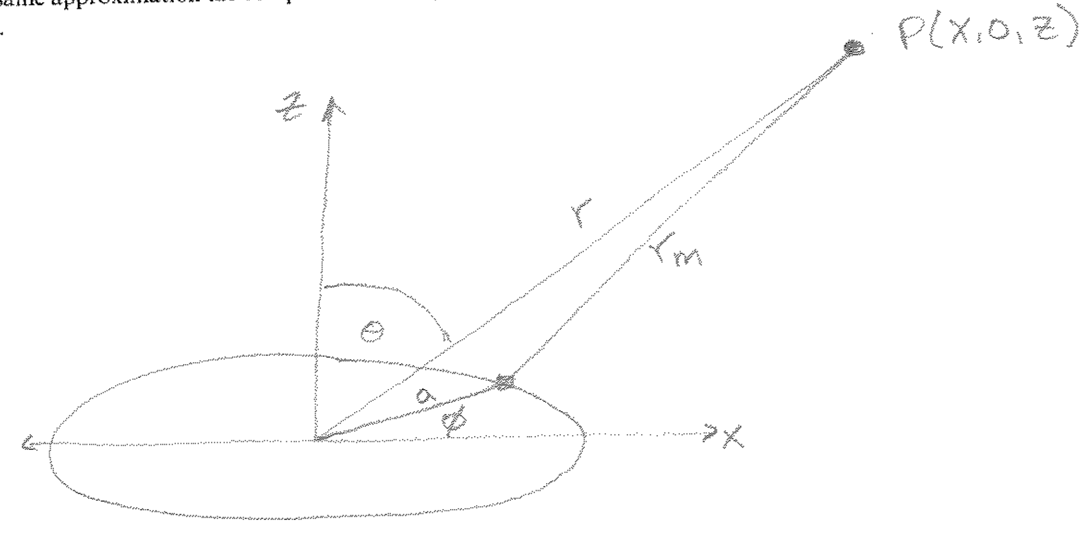
$$= g_0 + \frac{6}{4}\frac{MG}{a^2} - \frac{3}{4}\frac{MG}{a^2}$$

$$g = g_0 + \frac{3}{4}\frac{MG}{a^2}$$

$$g_0 - g = -\frac{GM_E}{R_E^2} + \frac{GM_E}{R_E^2} - \frac{3}{4}\frac{MG}{a^2}$$

$$\boxed{g_0 - g = -\frac{3}{4}\frac{MG}{a^2}}$$

6 - **15.** (a) Calculate the gravitational potential of a uniform circular ring of matter of radius a, mass M, at a distance r from the center of the ring in a direction making an angle θ with the axis of the ring. Assume that $r \gg a$, and calculate the potential only to second order in a/r.
b) Calculate to the same approximation the components of the gravitational field of the ring at the specified point.



a) U_g is independent of ϕ due to symmetry.

$$U_g = \int \frac{G\,dm}{|\vec{r}\,' - \vec{r}|}$$

$|\vec{r}_m| = |\vec{r}_a - \vec{r}|$ $\quad \vec{r}_a = a\cos\phi\,\hat{i} + a\sin\phi\,\hat{j}$

$\vec{r} = r\sin\theta\,\hat{i} + r\cos\theta\,\hat{k}$

$$= G\sigma a \int_0^{2\pi} \frac{d\phi}{[r^2+a^2-2ar\sin\theta\cos\phi]^{1/2}}$$

$$= \left[(a\cos\phi - r\sin\theta)^2 + a^2\sin^2\phi + r^2\cos^2\theta\right]^{1/2}$$

$$= 2G\sigma a \int_0^{\pi} [r^2+a^2-2ar\sin\theta\cos\phi]^{-1/2}\,d\phi$$

$$= \left[a^2(\cos^2\phi+\sin^2\phi) + r^2(\cos^2\theta+\sin^2\theta) - 2ra\sin\theta\cos\phi\right]^{1/2}$$

Binomial Expansion

$$(1+b)^n = 1 + nb + \frac{n(n-1)}{2!}b^2 + \frac{n(n-1)(n-2)}{3!}b^3 + \dots$$

$$|\vec{r}_m| = \left[r^2+a^2-2ra\sin\theta\cos\phi\right]^{1/2}$$

$$[r^2+a^2-2ar\sin\theta\cos\phi]^{-1/2} = \frac{1}{r}\left[1+\frac{a^2}{r^2} - \frac{2a}{r}\sin\theta\cos\phi\right]^{-1/2}$$

$dm = \sigma a\,d\phi$, σ = linear density

$$n = -\tfrac{1}{2} \quad b = \frac{a^2}{r^2} - \frac{2a}{r}\sin\theta\cos\phi$$

$$(1+b)^n = 1 - \frac{a^2}{2r^2} + \frac{a}{r}\sin\theta\cos\phi + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)4a^2}{2\,r^2}\sin^2\theta\cos^2\phi + \text{terms higher than } \frac{a^2}{r^2}$$

$$\simeq 1 - \frac{a^2}{2r^2} + \frac{a}{r}\sin\theta\cos\phi + \frac{3}{2}\frac{a^2}{r^2}\sin^2\theta\cos^2\phi$$

$$U_g = \frac{2G\sigma a}{r}\int_0^{\pi}\left[1-\frac{a^2}{2r^2}+\frac{a}{r}\sin\theta\cos\phi+\frac{3}{2}\frac{a^2}{r^2}\sin^2\theta\cos^2\phi\right]d\phi$$

$$\int_0^{\pi}\cos\phi\, d\phi = \sin\phi\Big|_0^{\pi} = 0$$

$$\int_0^{\pi}\cos^2\phi\, d\phi = \frac{1}{2}\int_0^{\pi}(1+\cos 2\phi)\,d\phi = \frac{1}{2}\left[\phi\Big|_0^{\pi}+\frac{1}{2}\sin 2\phi\Big|_0^{\pi}\right] = \frac{\pi}{2}$$

$$U_g = \frac{2G\sigma a}{r}\left[\pi-\frac{a^2}{2r^2}\pi+0+\frac{3}{2}\frac{a^2}{r^2}\sin^2\theta\,\frac{\pi}{2}\right] \qquad M = 2\pi\sigma a$$

$$= \frac{MG}{r}\left[1-\frac{a^2}{2r^2}+\frac{3}{4}\frac{a^2}{r^2}\sin^2\theta\right]$$

$$= \frac{MG}{r}\left[1-\frac{a^2}{2r^2}+\frac{3}{4}\frac{a^2}{r^2}(1-\cos^2\theta)\right]$$

$$= \frac{MG}{r}\left[1-\frac{a^2}{2r^2}+\frac{3a^2}{4r^2}-\frac{3a^2}{4r^2}\cos^2\theta\right]$$

$$= \frac{MG}{r}\left[1+\frac{a^2}{4r^2}-\frac{3a^2}{4r^2}\cos^2\theta\right]$$

$$= \frac{MG}{r}\left[1+\frac{a^2}{4r^2}(1-3\cos^2\theta)\right]$$

$$\boxed{U_g = \frac{MG}{r}+\frac{MGa^2}{4r^3}(1-3\cos^2\theta)}$$

b) Gradient in spherical coordinates

$$\vec{\nabla} = \vec{e}_1\frac{\partial}{\partial s_1}+\vec{e}_2\frac{\partial}{\partial s_2}+\vec{e}_3\frac{\partial}{\partial s_3}$$

$\vec{e}_1 = \hat{r} \quad ds_1 = dr \qquad \vec{e}_2 = \hat{\theta} \quad ds_2 = r\,d\theta \qquad \vec{e}_3 = \hat{\phi} \quad ds_3 = r\sin\theta\, d\phi$

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\hat{\phi}$$

$\vec{g}$ is independent of ϕ

$$\vec{g}_r = \frac{\partial U}{\partial r}\hat{r}$$

$$\boxed{\vec{g}_r = \left[-\frac{MG}{r^2} - \frac{3MGa^2}{r^4}(1-3\cos^2\theta)\right]\hat{r}}$$

$$\vec{g}_\theta = \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\theta}$$

$$= \frac{1}{r}\,\frac{6MGa^2}{4r^3}\cos\theta\sin\theta\,\hat{\theta}$$

$\sin(A+B) = \sin A\cos B + \cos A\sin B$

$\sin(2A) = 2\sin A\cos A$

$$\boxed{\vec{g}_\theta = \frac{3MGa^2}{4r^4}\sin 2\theta\,\hat{\theta}}$$

7-1

a) Fixed coordinate system:

$$m\frac{d^2\vec{x}}{dt^2} = \vec{F}$$

Coordinate system translating with acceleration $\vec{g}$:

$$m\frac{d^2\vec{x}^*}{dt^2} + m\vec{a}_h = \vec{F}$$

$$\vec{a}_h = \vec{g}$$

$$m\frac{d^2\vec{x}^*}{dt^2} + m\vec{g} = \vec{F}$$

$$m\frac{d^2\vec{x}^*}{dt^2} = \vec{F} - m\vec{g}$$

$$\vec{F} = m\vec{g}$$

$$\frac{d^2x^*}{dt^2} = \vec{g} - \vec{g}$$

$$\frac{d^2x^*}{dt^2} = 0 = a_x^*$$

$$\frac{dx^*}{dt} = C_1$$

$$v_0^* = 0 \rightarrow \frac{dx^*}{dt} = 0 \rightarrow C_1 = 0$$

$$x^* = C_2$$

$$x_0^* = 0 \rightarrow C_2 = 0$$

$$\boxed{x^* = 0}$$

$$a_x = a_x^* + a_{h_x} \qquad a_{h_x} = -g$$

$$a_x = 0 - g$$

$$ma_x = -mg$$

$$a_x = -g$$

$$\frac{dx^2}{dt} = -g \rightarrow \underset{v_0=0}{\frac{dx}{dt} = -gt + C_1} \rightarrow \frac{dx}{dt} = -gt \rightarrow \underset{x_0=0}{x = -\frac{1}{2}gt^2 + C_2}$$

$$\boxed{x = -\frac{1}{2}gt^2}$$

b) $\vec{a}_h = \vec{g}$ $\quad m\ddot{x} + b\dot{x} + kx = F$

$F = m\ddot{x} + b\dot{x}$, $k=0$, $F = mg$, $a_h = g$

$\ddot{x} = \ddot{x}^* - a_h$

air is fixed

$\dot{x}^* = \dot{x} - \frac{dh}{dt}$ $\quad \frac{d^2h}{dt} = g$

$\dot{x}^* = \dot{x} - gt$ $\quad \frac{dh}{dt} = gt + C_1$

$\dot{x} = \dot{x}^* + gt$ $\quad \frac{dh}{dt}_0 = 0 \rightarrow C_1 = 0$

$mg = m(\ddot{x}^* - a_h) + b(\dot{x}^* + gt)$

$= m\ddot{x}^* - mg + b(\dot{x}^* + gt)$

$m\ddot{x}^* = -b(\dot{x}^* + gt)$

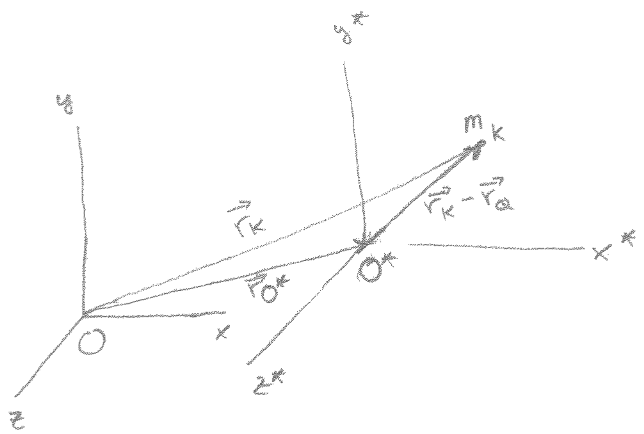
$$\boxed{ma^* = -bv^* - bgt}$$

7-3

Eq. (5-5) $\frac{d\vec{L}}{dt} = \vec{N}$

where $\vec{L}$ and $\vec{N}$ are about the point $\vec{R}$

$$M\vec{R} = \sum_{k=1}^{N} m_k \vec{r}_k$$



$$a_h = \frac{d^2\vec{r}_{O^*}}{dt^2} = \frac{d^2\vec{h}}{dt^2} = \ddot{\vec{r}}_{O^*}$$

$$\vec{r} = \vec{r}_k$$

$$\vec{r}^* = \vec{r}_k - \vec{r}_{O^*}$$

(7.38) $\vec{r} = \vec{r}^* + \vec{h}$

(7.39) $\frac{d\vec{r}}{dt} = \frac{d^*\vec{r}^*}{dt} + \vec{\omega} \times \vec{r}^* + \frac{d\vec{h}}{dt}$

(7.40) $\frac{d^2\vec{r}}{dt^2} = \frac{d^{*2}\vec{r}^*}{dt^2} + \vec{\omega} \times (\vec{\omega} \times \vec{r}^*) + 2\vec{\omega} \times \frac{d^*\vec{r}^*}{dt} + \frac{d\vec{\omega}}{dt} \times \vec{r}^* + \frac{d^2\vec{h}}{dt^2}$

$$\vec{L}_{kO^*} = m_k(\vec{r}_k - \vec{r}_{O^*}) \times (\dot{\vec{r}}_k - \dot{\vec{r}}_{O^*})$$

$$\frac{d\vec{L}_{kO^*}}{dt} = (\vec{r}_k - \vec{r}_{O^*}) \times \frac{d\vec{p}_k}{dt} + m_k(\dot{\vec{r}}_k - \dot{\vec{r}}_{O^*}) \times (\dot{\vec{r}}_k - \dot{\vec{r}}_{O^*}) - m(\vec{r}_k - \vec{r}_{O^*}) \times \ddot{\vec{r}}_{O^*}$$

$$= (\vec{r}_k - \vec{r}_{O^*}) \times \vec{F}_k^e + (\vec{r}_k - \vec{r}_{O^*}) \times \vec{F}_k^i - m_k(\vec{r}_k - \vec{r}_{O^*}) \times \ddot{\vec{r}}_{O^*}$$

$$\vec{L}_{O^*} = \sum_{k=1}^{N} \vec{L}_{kO^*}, \quad \vec{N}_{O^*} = \sum_{k=1}^{N} (\vec{r}_k - \vec{r}_{O^*}) \times \vec{F}_k^e$$

$\frac{d\vec{L}_{O^*}}{dt}$ $\vec{N}_{O^*} - m_k(\vec{r}_k - \vec{r}_{O^*}) \times \ddot{\vec{r}}_{O^*}$ { internal torque is zero }

$$= \vec{N}_{O^*} - M(\vec{R}^* - \vec{r}_{O^*}) \times \ddot{\vec{r}}_{O^*}$$

$$\ddot{\vec{r}}_{o^*} = a_h = \frac{d^2\vec{h}}{dt^2}$$

From Eq. (7.40)

$$a_h = \frac{d^2\vec{h}}{dt^2} = \frac{d^2\vec{r}}{dt^2} - \frac{d^{*2}\vec{r}^*}{dt^2} - \vec{\omega}\times(\vec{\omega}\times\vec{r}^*) - 2\vec{\omega}\times\frac{d^*\vec{r}^*}{dt} - \frac{d\vec{\omega}}{dt}\times\vec{r}^*$$

$$\boxed{\frac{d\vec{L}}{dt} = \vec{N}_{o^*} - M(\vec{R}^* - \vec{r}_o^*)\times\left[\frac{d^2\vec{r}}{dt^2} - \frac{d^{*2}\vec{r}^*}{dt^2} - \vec{\omega}\times(\vec{\omega}\times\vec{r}^*) - 2\vec{\omega}\times\frac{d^*\vec{r}^*}{dt} - \frac{d\vec{\omega}}{dt}\times\vec{r}^*\right]}$$

7-5

$\underrightarrow{v, \ell}$

$$\vec{f} = \vec{\nabla} p = \rho \vec{g}$$

Introducing the centrifugal and Coriolos forces in the rotating coordinate system can be treated as a fixed coordinate system

$$m \frac{d^{*2}\vec{r}}{dt} = \vec{F} + m\vec{g}_e - 2m\vec{\omega} \times \frac{d^{*}\vec{r}}{dt}$$

$$\frac{d^{*}r}{dt} = v\hat{x} \quad \text{(westerly wind with velocity } v\text{)}$$

$$m\frac{d^{*2}\vec{r}}{dt} = 0 \quad \text{(constant velocity)}$$

$$0 = \vec{F} - 2m\vec{\omega} \times v\hat{x}$$

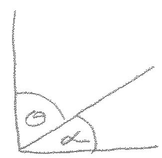
$$\vec{F} = 2m\omega v \sin\alpha$$

$$\vec{F} = \rho\vec{g}V$$

$$\frac{\vec{F}}{V} = \rho\vec{g} = \vec{\nabla} p$$

$$\vec{\nabla} p = 2\frac{m}{V}\omega v \sin\alpha$$

$$= 2\rho\omega v \sin\alpha \qquad \sin\alpha = \cos\theta$$



$$\boxed{\vec{\nabla} p = 2\rho\omega v \cos\theta}$$

$$\rho_{air} = 1.2 \frac{kg}{m^3} \qquad \omega_e = \frac{2\pi}{24 hrs} = 7.3 \times 10^{-5} \frac{rad}{s} \qquad v = 15\frac{m}{s} \text{ (10 miles per hour)}$$

$$\nabla p = 2\left(1.2\frac{kg}{m^3}\right)\left(7.3\times10^{-5}/s\right)\left(15\frac{m}{s}\right)$$

$$= .002 \frac{kg}{m^3}\frac{m}{s^2}$$

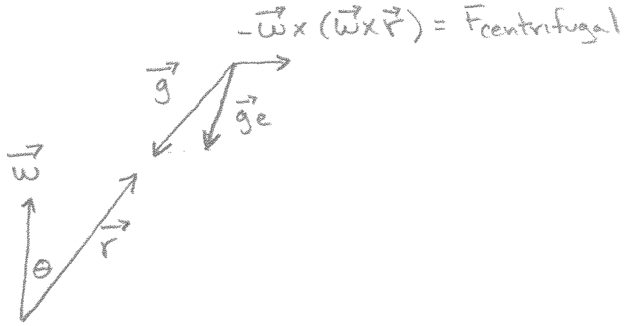
$$= .002 \frac{N}{m^3} \qquad 2.2\,lb = 1\,kg = 9.81\,N$$

$$= .002\frac{N}{m^3} \times \frac{2.2\,lb}{9.81\,N} \times \frac{(.0254\,m)^3}{1\,in^3} \times \frac{12\,in}{1\,ft} \times \frac{5280\,ft}{1\,mile}$$

$$\boxed{\nabla p = 4.7\times10^{-4}\frac{lb}{in^2\text{-}mile}}$$

7-7

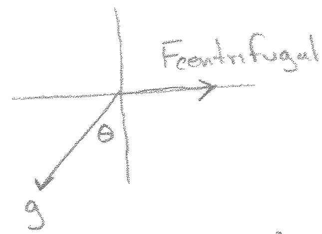
a) $\vec{F}_{cor} = 2m\vec{\omega} \times \frac{d^*\vec{r}}{dt}$



$|\vec{\omega} \times \vec{r}| = \omega r \sin\theta \rightarrow |\vec{\omega} \times (\vec{\omega} \times \vec{r})| = \omega^2 r \sin\theta$

$\vec{g}_e = \vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{r})$

$\vec{g}_e = \vec{g} - \omega^2 r \sin\theta\, \hat{x}$



$\vec{g}_x = g\sin\theta\,\hat{x}$

$\vec{g}_y = g\cos\theta\,\hat{y}$

To a first approximation

$\vec{g}_e \approx \vec{g}$

$\frac{d^{*2}\vec{r}}{dt^2} = \vec{g}$

$\frac{d^*\vec{r}}{dt} = \vec{g}t$

$F_{cor}(t) = 2m\vec{\omega} \times \vec{g}t$

$\boxed{F_{cor}(t) = 2m\omega g t \sin\theta, \text{towards the east}}$

b) $F_{cor}(t) = 2m\omega g t \sin\theta$

$h = \frac{1}{2}gt^2$

$t^2 = \frac{2h}{g}$

$F_{cor} = 2mg\omega\left(\frac{2h}{g}\right)^{1/2}\sin\theta$

$ma_{cor} = 2^{3/2} m\omega (gh)^{1/2}\sin\theta$

$a_{cor} = 2^{3/2}\omega (gh)^{1/2}\sin\theta$

$d = \frac{1}{2}a_{cor}t^2$

$= \frac{1}{2}a_{cor}\frac{2h}{g}$

$d = a_{cor}\frac{h}{g}$

$d_1 = 2^{3/2}\omega\frac{h_1^{3/2}}{g^{1/2}}\sin\theta \qquad d'(h) = \frac{2^{3/2}\omega}{g^{1/2}}\frac{3h^{1/2}}{2}\sin\theta$

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$h_{n+1} = h_1 - \frac{h_1^{3/2}}{\frac{3}{2}h_1^{1/2}}$

$h_2 = h_1 - \frac{2}{3}h_1$

$= \frac{1}{3}h_1$

$d_2 = a_{cor}\frac{h_2}{g}$

$d_2 = a_{cor}\frac{h_1}{3g}$

$\boxed{d_2 = \frac{1}{3}d_1 = \frac{2^{3/2}\omega h^{3/2}}{3g^{1/2}}\sin\theta = \left(\frac{8\omega^2 h^3}{9g}\right)^{1/2}\sin\theta}$

Alternate method for b)

$$F_{cor} = 2m\omega g t \sin\theta$$

$$a_{cor} = 2\omega g t \sin\theta$$

$$v = \int a_{cor}\,dt = 2\omega g \sin\theta \int t\,dt$$

$$v = \omega g \sin\theta\, t^2$$

$$x = \int v\,dt = \omega g \sin\theta \int t^2\,dt$$

$$x = \frac{1}{3}\omega g t^3 \sin\theta$$

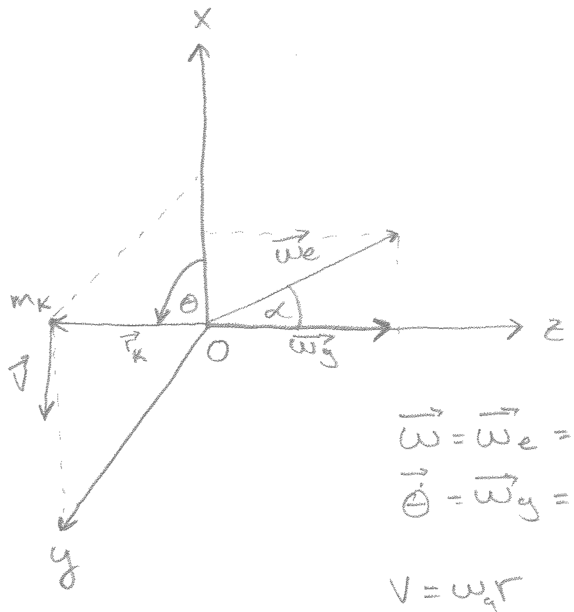
$$h = \frac{1}{2} g t^2$$

$$t = \left(\frac{2h}{g}\right)^{1/2}$$

$$x = \frac{1}{3}\omega g \left(\frac{2h}{g}\right)^{3/2} \sin\theta$$

$$\boxed{x = \left(\frac{8\omega^2 h^3}{9g}\right)^{1/2} \sin\theta}$$

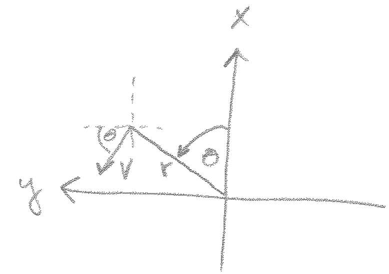
7-11



$$\vec{\omega} = \vec{\omega}_e = \omega \sin\alpha\, \hat{i} + \omega\cos\alpha\, \hat{k}$$

$$\vec{\dot{\theta}} = \vec{\omega}_g = \dot{\theta}\, \hat{k}$$

$$V = \omega_g r$$
$$= \dot{\theta} r$$



$$\vec{V} = -\dot{\theta} r \sin\theta\, \hat{i} + \dot{\theta} r\cos\theta\, \hat{j}$$

$$\vec{F}_{cor_{m_K}} = -2m_K \vec{\omega}_{net} \times \vec{V}$$

$$\vec{\omega}_{net} = \vec{\omega} + \vec{\omega}_g$$
$$= \omega\sin\alpha\, \hat{i} + (\omega\cos\alpha + \dot{\theta})\hat{k}$$

$$= -2m_K \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \omega\sin\alpha & 0 & \omega\cos\alpha + \dot{\theta} \\ -\dot{\theta} r\sin\theta & \dot{\theta} r\cos\theta & 0 \end{vmatrix}$$

$$= -2m_K\left[-\dot{\theta} r\cos\theta(\omega\cos\alpha + \dot{\theta})\hat{i} - \dot{\theta} r\sin\theta(\omega\cos\alpha+\dot{\theta})\hat{j} + \omega\dot{\theta} r\cos\theta\sin\alpha\, \hat{k}\right]$$

$$\vec{N}_{cor_{m_K}} = \vec{r}_K \times \vec{F}_{cor_{m_K}} \qquad \vec{r}_K = r\cos\theta\, \hat{i} + r\sin\theta\, \hat{j}$$

$$= -2m_K \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ r\cos\theta & r\sin\theta & 0 \\ -\dot{\theta} r\cos\theta(\omega\cos\alpha+\dot{\theta}) & -\dot{\theta} r\sin\theta(\omega\cos\alpha+\dot{\theta}) & \omega\dot{\theta} r\cos\theta\sin\alpha \end{vmatrix}$$

$$\vec{N}_{cor_{m_k}} = -2m_k\Big[\omega\dot{\theta}r^2\sin\theta\cos\theta\sin\alpha\,\hat{i} + \omega\dot{\theta}r^2\cos^2\theta\,\sin\alpha\,\hat{j} + \left(-\dot{\theta}r^2\sin\theta\cos\theta(\omega\cos\alpha+\dot{\theta}) + \dot{\theta}r^2\sin\theta\cos\theta(\omega\cos\alpha+\dot{\theta})\right)\hat{k}\Big]$$

$$= -2m_k\left[\omega\dot{\theta}r^2\sin\theta\cos\theta\sin\alpha\,\hat{i} + \omega\dot{\theta}r^2\cos^2\theta\sin\alpha\,\hat{j}\right]$$

$$\boxed{\vec{N}_{cor_{m_k}} = -2m_k\omega\dot{\theta}r^2\sin\alpha\left[\sin\theta\cos\theta\,\hat{i} + \cos^2\theta\,\hat{j}\right]}$$

$$\vec{N}_{cor} = \sum_{K=1}^{N}\vec{N}_{cor_{mK}}$$

$$= \int_0^{2\pi}\frac{\vec{N}_{cor_{mK}}}{mK}\lambda r\,d\theta \qquad \lambda = \text{linear mass density}$$

$$= -2\lambda r^3\omega\dot{\theta}\sin\alpha\left[\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\,\hat{i} + \int_0^{2\pi}\cos^2\theta\,d\theta\,\hat{j}\right]$$

$$= -2\lambda r^3\omega\dot{\theta}\sin\alpha\left[\frac{\sin^2\theta}{2}\Big|_0^{2\pi}\hat{i} + \int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)\,d\theta\,\hat{j}\right]$$

$$= -2\lambda r^3\omega\dot{\theta}\sin\alpha\left[\pi + \frac{1}{4}\sin 2\theta\Big|_0^{2\pi}\right]\hat{j}$$

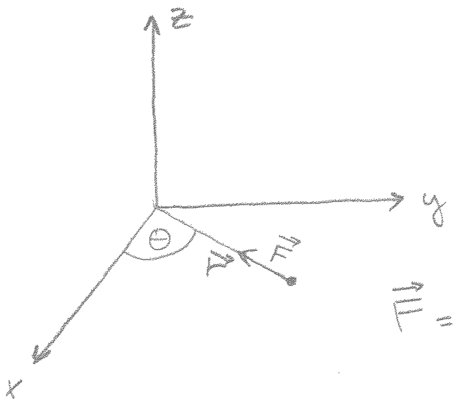
$$= -2\lambda\pi r^3\omega\dot{\theta}\sin\alpha\,\hat{j}$$

$$M = 2\pi r\lambda$$

$$\boxed{\vec{N}_{cor} = -\hat{j}\,Mr^2\omega\dot{\theta}\sin\alpha}$$

Note: answer in text is $-\frac{1}{2}\hat{j}\,Mr^2\omega\dot{\theta}\sin\alpha$

7-13



$\vec{F} = -k\vec{r}$ (central force)

Introducing a rotating coordinate system about the z-axis such that $m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \vec{F}$ {centrifugal force cancels F}.

With respect to the rotating coordinate system:

$$m\frac{d^{*2}\vec{r}}{dt^2} = \vec{F} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - 2m\vec{\omega}\times\frac{d^*\vec{r}}{dt}$$

Now,

$\vec{\omega} = \omega\hat{z}^*$

$\vec{r} = x\hat{x}^* + y\hat{y}^*$

$$\vec{\omega}\times\vec{r} = \begin{vmatrix} \hat{x}^* & \hat{y}^* & \hat{z}^* \\ 0 & 0 & \omega \\ x & y & 0 \end{vmatrix}$$

$$= -y\omega\hat{x}^* + x\omega\hat{y}^*$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \begin{vmatrix} \hat{x}^* & \hat{y}^* & \hat{z}^* \\ 0 & 0 & \omega \\ -y\omega & x\omega & 0 \end{vmatrix}$$

$$= -x\omega^2\hat{x}^* - y\omega^2\hat{y}^*$$

$$= -\omega^2(x\hat{x}^* + y\hat{y}^*)$$

$$\vec{\omega}\times(\vec{\omega}\times\vec{r}) = -\omega^2\vec{r}$$

But, $m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = \vec{F}$

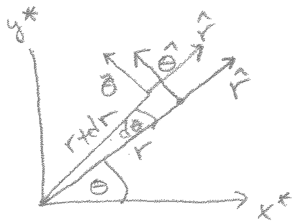
$$-m\omega^2\vec{r} = -k\vec{r}$$

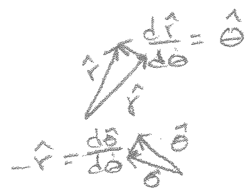
$$\omega^2 = \frac{k}{m} \rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}}$$

Since $\vec{F}$ cancels $-m\vec{\omega}\times(\vec{\omega}\times\vec{r})$,

$$m\frac{d^{*2}\vec{r}}{dt^2} = -2m\vec{\omega}\times\frac{d^*\vec{r}}{dt}$$

And since $\vec{r}$ lies in the x-y plane, $\frac{d^*\vec{r}}{dt}$ lies in the same plane, thus, $\vec{\omega}$ and $\frac{d^*\vec{r}}{dt}$ are always $\perp$.





$$\vec{r} = r(\theta)\hat{r}$$

$$\frac{d^*\vec{r}}{dt} = \frac{dr}{dt}\hat{r}^* + r\frac{d\hat{r}^*}{dt}$$

$$= \frac{dr}{dt}\hat{r}^* + r\frac{d\hat{r}^*}{d\theta}\frac{d\theta}{dt}$$

$$= \dot{r}\hat{r}^* + r\dot{\theta}\hat{\theta}^*$$

$$-2m\vec{\omega}\times\frac{d^*\vec{r}}{dt} = -2m\begin{vmatrix} \hat{r}^* & \hat{\theta}^* & \hat{z}^* \\ 0 & 0 & \omega \\ \dot{r} & r\dot{\theta} & 0 \end{vmatrix}$$

$$= -2m\left[-r\omega\dot{\theta}\hat{r}^* + \dot{r}\omega\hat{\theta}^*\right]$$

$$= 2mr\dot{\theta}\omega\hat{r}^* - 2m\dot{r}\omega\hat{\theta}^*$$

$$\frac{d^{*2}\vec{r}}{dt^2} = \frac{d^*}{dt}\left(\frac{d^*\vec{r}}{dt}\right)$$

$$= \frac{d^*}{dt}\left(\dot{r}\hat{r}^* + r\dot{\theta}\hat{\theta}^*\right)$$

$$= \ddot{r}\hat{r}^* + \dot{r}\frac{d^*\hat{r}^*}{dt} + r\dot{\theta}\frac{d^*\hat{\theta}^*}{dt} + \left[r\ddot{\theta} + \dot{r}\dot{\theta}\right]\hat{\theta}^*$$

$$= \ddot{r}\hat{r}^* + \dot{r}\frac{d^*\hat{r}^*}{d\theta}\cdot\frac{d\theta}{dt} + r\dot{\theta}\frac{d^*\hat{\theta}^*}{d\theta}\frac{d\theta}{dt} + \left[r\ddot{\theta} + \dot{r}\dot{\theta}\right]\hat{\theta}^*$$

$$= \ddot{r}\hat{r}^* + \dot{r}\dot{\theta}\hat{\theta}^* - r\dot{\theta}^2\hat{r}^* + r\ddot{\theta}\hat{\theta}^* + \dot{r}\dot{\theta}\hat{\theta}^*$$

$$= \left(\ddot{r} - r\dot{\theta}^2\right)\hat{r}^* + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\theta}^*$$

Since $\vec{F}$ and $\vec{F}_{centrifugal}$ just cancel each other, there is no acceleration in the $\hat{r}^*$ direction. So,

$$\frac{d^{*2}\vec{r}}{dt^2} = (2\dot{r}\dot{\theta} + r\ddot{\theta})\,\hat{\theta}^*$$

$$\frac{d^{*2}\vec{r}}{dt^2} = r\ddot{\theta}\,\hat{\theta}^*$$

$$-2m\vec{\omega} \times \frac{d^*\vec{r}}{dt} = 2mr\dot{\theta}\,\hat{r}^*$$

So,

1) $mr\ddot{\theta}\,\hat{\theta}^* = -2m\dot{r}\omega\,\hat{\theta}^*$

2) $mr\dot{\theta}^2\,\hat{r}^* = 2mr\dot{\theta}\omega\,\hat{r}^*$ $\qquad \dot{r} = \text{constant} = C_0$

1) $mr\ddot{\theta} = -2mC_0\omega$

2) $mr\dot{\theta}^2 = 2mr\dot{\theta}\omega$

1) $\ddot{\theta} = -2\frac{C_0}{r}\omega$

$\frac{d}{dt}\dot{\theta} = -\frac{2C_0\omega}{r}$

$\dot{\theta} = -\frac{2C_0\omega}{r}t + C_1$

$\theta = -\frac{C_0\omega}{r}t^2 + C_1 t + C_2$

$\theta(0) = \theta_0$

$\theta = -\frac{C_0\omega}{r}t^2 + C_1 t + \theta_0$

2) $-r\dot{\theta}^2 - 2r\omega\dot{\theta} = 0$

$\dot{\theta}^2 + 2\omega\dot{\theta} = 0$

$\dot{\theta}(\dot{\theta} + 2\omega) = 0$

$\dot{\theta} = 0$ or $\dot{\theta} = -2\omega$

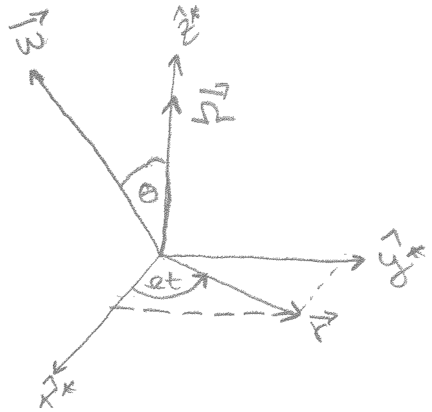
$\theta = \text{Constant}$ or $\theta = -2\omega t + \theta_0$

Therefore, the motion is circular where r and $\dot{\theta}$ depend on the initial conditions. The mass, m, has an angular velocity of -2ω.

7-15

Eq. 7.46: $m\dfrac{d^{*2}\vec{r}}{dt^2} = \vec{\tau} + m\vec{g}_e - 2m\vec{\omega}\times\dfrac{d^*\vec{r}}{dt}$

Let the circular motion of the bob have an angular frequency Ω. Then



$\hat{z}^*$ is $\perp$ to earth surface

θ is colatitude

$$\vec{r} = r\cos(\Omega t)\hat{x}^* + r\sin(\Omega t)\hat{y}^*$$

$$\frac{d^*\vec{r}}{dt} = -r\Omega\sin(\Omega t)\hat{x}^* + r\Omega\cos(\Omega t)\hat{y}^*$$

$$\vec{F}_{cor} = -2m\vec{\omega}\times\frac{d^*\vec{r}}{dt}$$

$$= -2m\begin{vmatrix} \hat{x}^* & \hat{y}^* & \hat{z}^* \\ 0 & -\omega\sin\theta & \omega\cos\theta+\Omega \\ -r\Omega\sin(\Omega t) & r\Omega\cos(\Omega t) & 0 \end{vmatrix}$$

$$= -2m\left(-r\Omega\cos(\Omega t)[\omega\cos\theta+\Omega]\hat{x}^* - r\Omega\sin(\Omega t)[\omega\cos\theta+\Omega]\hat{y}^* + r\omega\Omega\sin\theta\sin(\Omega t)\hat{z}^*\right)$$

$$\vec{F}_{cor_{vert}} = -2mr\omega\Omega\sin\theta\sin(\Omega t)\hat{z}^* \quad \left\{\Omega = \frac{2\pi}{T}\right\}$$

The time average of $\vec{F}_{cor_{vert}}$ is

$$\frac{1}{T}\int_0^T F_{cor_{vert}}\,dt = \frac{-2mr\omega\Omega\sin\theta}{T}\int_0^T \sin\left(\frac{2\pi}{T}t\right)dt$$

$$= \frac{2mr\omega\Omega\sin\theta}{T}\left[\frac{T}{2\pi}\cos\left(\frac{2\pi}{T}t\right)\Big|_0^T\right]$$

$$= \frac{mr\omega\Omega\sin\theta}{\pi}[1-1]$$

$$= 0, \text{ and } |\vec{F}_{cor_v}| \ll 1 \quad \{\omega \doteq 10^{-5}\,\text{rad/s}\}$$

Therefore, the vertical component of the Coriolis force can be neglected.
The horizontal component is

$$\vec{F}_{cor_H} = 2mr\Omega\left[\cos(\Omega t)\,\hat{x}^* + \sin(\Omega t)\,\hat{y}^*\right]\left[\omega\cos\theta + \Omega\right]$$

$$= 2mr\Omega\omega\cos\theta\left[\cos(\Omega t)\,\hat{x}^* + \sin(\Omega t)\,\hat{y}^*\right] + 2mr\Omega^2\left[\cos(\Omega t)\,\hat{x}^* + \sin(\Omega t)\,\hat{y}^*\right]$$

7-17

Problem 7-16 must be solved first.

7-16 For Larmor's Theorem to apply,

$$\omega_c >> \omega_L$$

To find ω_c, the angular frequency due to the central force, use Keplar's Law to relate the period to the semi major axis

$$\tau^2 = 4\pi^2 a^3 \left|\frac{m}{K}\right|$$

$a = 10^{-8}\,cm = 10^{-10}\,m$

Gaussian units:

$K = q^2 = (ce)^2 = (4.8 \times 10^{-10})^2$

$c = 2.99 \times 10^8\,m/s \times 2.99 \times 10^9\,dm/s$

$m_e = 9.1 \times 10^{-31}\,kg = 9.1 \times 10^{-34}\,g$

$$\tau^2 = 4\pi^2 10^{-30} \left|\frac{9.1 \times 10^{-31}}{(9 \times 10^9)(1.6 \times 10^{-19})^2}\right|$$

$$= 1.6 \times 10^{-31}\,s^2$$

$$\tau = 4 \times 10^{-16}\,s$$

$$\omega = \frac{2\pi}{\tau}$$

$$\omega_c = 1.6 \times 10^{16}\,rad/s$$

$$\omega_L = -\frac{q}{2m} B$$

$B = 10{,}000\,G = 1\,T$ {MKS units}

$$= \frac{1.6 \times 10^{-19}}{2(9.1 \times 10^{-31})}(1)$$

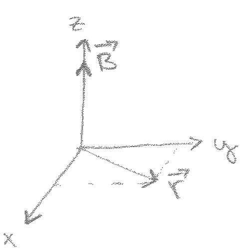
$$\omega_L = 8.8 \times 10^{10}\,rad/s$$

Therefore, Larmor's Theorem applies and the angular velocity of precession is

$$\boxed{\omega_L = 8.8 \times 10^{10}\,rad/s}$$

The potential for the last term of Eq. (7.58) when $\vec{B}$ is $\perp$ to the plane of the orbit is, starting with

$$\frac{q^2}{4mc^2} \vec{B} \times (\vec{B} \times \vec{r}) = -\frac{q^2 B^2 r}{4mc^2} \hat{r}$$



$$F_1(r) = -\frac{K}{r^2}$$

$$V_1(r) = -\int_0^r F_1(r)dr$$

$$= -\frac{K}{r}$$

$$V_2(r) = -\int_0^r F_2(r)dr$$

$$= \frac{q^2B^2}{4mc^2}\int_0^r r\,dr$$

$$= \frac{q^2B^2}{4mc^2}\frac{r^2}{2}$$

$$'V'(r) = V_1(r) + V_2(r) + \frac{L^2}{2mr^2}$$

$$= -\frac{k}{r} + \frac{q^2B^2}{4mc^2}\frac{r^2}{2} + \frac{L^2}{2mr^2}$$

$$\frac{dV'}{dr} = \frac{K}{r^2} + \frac{q^2B^2}{4mc^2}r - \frac{L^2}{mr^3}$$

$$0 = Kr_0 + \frac{q^2B^2}{4mc^2}r_0^4 - \frac{L^2}{m}$$

$$\frac{d^2V'}{dr^2} = -\frac{2K}{r^3} + \frac{q^2B^2}{4mc^2} + \frac{3L^2}{mr^4}$$

$$\omega_r^2 = \frac{1}{m}\left(\frac{d^2V'}{dr^2}\right)_{r_0} = \frac{1}{m}\left(\frac{-2K}{r_0^3} + \frac{q^2B^2}{4mc^2} + \frac{3L^2}{mr_0^4}\right)$$

$$L^2 = mKr_0 + \frac{q^2B^2}{4c^2}r_0^4$$

$$\omega_r^2 = \frac{1}{m}\left(-\frac{2K}{r_0^3} + \frac{q^2B^2}{4mc^2} + \frac{3}{mr_0^4}\left[mKr_0 + \frac{q^2B^2}{4c^2}r_0^4\right]\right)$$

$$= \frac{1}{m}\left(-\frac{2K}{r_0^3} + \frac{q^2B^2}{4mc^2} + \frac{3K}{r_0^3} + \frac{3q^2B^2}{4mc^2}\right)$$

$$= \frac{1}{m}\left(\frac{K}{r_0^3} + \frac{q^2B^2}{4mc^2} + \frac{3q^2B^2}{4mc^2}\right)$$

$$\omega_r = \left(\frac{K}{mr_0^3} + \frac{q^2B^2}{4m^2c^2} + \frac{3q^2B^2}{4m^2c^2}\right)^{1/2}$$

$$= \left(\frac{K}{mr_0^3} + 4\omega_L^2\right)^{1/2}$$

$$\approx \left(\frac{K}{mr_0^3}\right)^{1/2} + \frac{1}{2}\left(\frac{mr_0^3}{K}\right)^{1/2}\left[4\omega_L^2\right]$$

$$(a+b)^{1/2} = a^{1/2} + \frac{1}{2}a^{-1/2}b + \cdots$$

$$L = mr_0^2\dot{\theta}$$

$$mr_0^2\dot{\theta} = \left(mKr_0 + \frac{q^2B^2}{4c^2}r_0^4\right)^{1/2}$$

$$\dot{\theta} = \left(\frac{K}{mr_0^3} + \frac{q^2B^2}{4m^2}\right)^{1/2}$$

$$\dot{\theta} = \left(\frac{K}{mr_0^3} + \omega_L^2\right)^{1/2}$$

$$\dot{\theta} \approx \left(\frac{K}{mr_0^3}\right)^{1/2} + \frac{1}{2}\left(\frac{mr_0^3}{K}\right)^{1/2}\omega_L^2$$

$\omega_p = \omega_r - \dot{\Theta}$

$= \left(\frac{K}{mr_0^3}\right)^{1/2} + \frac{1}{2}\left(\frac{mr_0^3}{K}\right)^{1/2}(4\omega_L^2) - \left[\left(\frac{K}{mr_0^3}\right)^{1/2} + \frac{1}{2}\left(\frac{mr_0^3}{K}\right)^{1/2}(\omega_L^2)\right]$

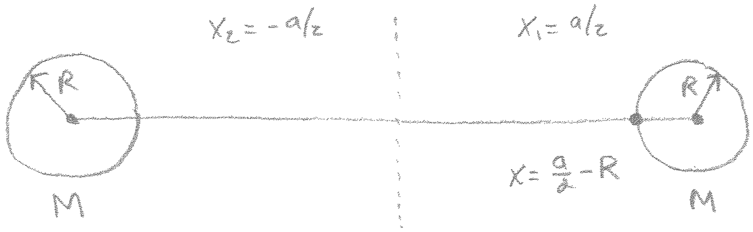
$= -\frac{1}{2}\left(\frac{mr_0^3}{K}\right)^{1/2}(3\omega_L^2)$

$= -\frac{1}{2}\left[\frac{(9.1\times10^{-31})(10^{-10})^3}{(9\times10^9)(1.6\times10^{-19})^2}\right]^{1/2}\left(3(8.8\times10^{10})^2\right)$

$\boxed{\omega_p = -.73\times10^6 \text{ rad/s},}$

Therefore, ω_p should be subtracted from ω_L when the electron orbital angular frequency vector is in the same direction as $\vec{B}$, and added when the frequency vector is anti-parallel to $\vec{B}$.

7-21



$$V'_{y=0} = \frac{-mMG}{\left[(x-x_1)^2\right]^{1/2}} - \frac{mMG}{\left[(x-x_2)^2\right]^{1/2}} - \frac{m(M+M)Gx^2}{2a^3}$$

To escape the planet system

$$v = \pm\sqrt{\frac{2}{m}}\left(E - V(x)\right)^{1/2}, \quad E = 0$$

$$= \pm\sqrt{\frac{2}{m}}\left(-V(x)\right)^{1/2}$$

$$V' = \frac{-mMG}{\left[\left(\frac{a}{2} - R - \frac{a}{2}\right)^2\right]^{1/2}} - \frac{mMG}{\left[\left(\frac{a}{2} - R + \frac{a}{2}\right)^2\right]^{1/2}} - \frac{mMG}{a^3}\left(\frac{a}{2} - R\right)^2$$

$$V' = -mMG\left[\frac{1}{R} + \frac{1}{a-R} + \frac{\left(\frac{a}{2} - R\right)^2}{a^3}\right]$$

$$v = \sqrt{\frac{2}{m}}\left(mMG\right)^{1/2}\left(\frac{1}{R} + \frac{1}{a-R} + \frac{(a-2R)^2}{4a^3}\right)^{1/2}$$

The effective potential, V', at $x = 0$ is

$$V'_{x=0} = \frac{-mMG}{\frac{a}{2}} - \frac{mMG}{\frac{a}{2}}$$

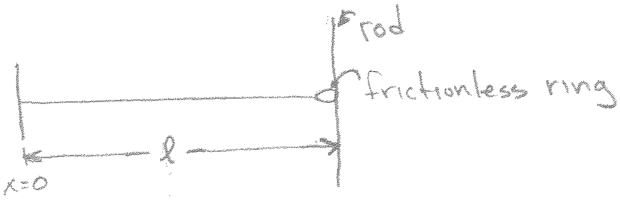
$$= \frac{-4mMG}{a}$$

Therefore, $V'_{x=0}$ is subtracted from the planet system escape potential so that the rocket remains in the system and travels from one planet to the other. So,

$$\boxed{v = \sqrt{2MG}\left[\frac{1}{R} + \frac{1}{a-R} - \frac{4}{a} + \frac{(a-2R)^2}{4a^3}\right]^{1/2}}$$

If the system is non-rotating, the centrifugal term would be dropped thereby showing the escape velocity would be less compared to the rotational system.

8-1



a) Determine boundary condition at $x=\ell$

The vertical component of string tension, τ, is

$$\tau_v = \tau \frac{\partial u}{\partial x}$$

Since the sliding ring is frictionless at $x=\ell$ there is no vertical force present.

So,

$$0 = \tau_v = \tau \frac{\partial u}{\partial x}$$

Since $\tau \neq 0$ then

$$\boxed{\left[\frac{\partial u}{\partial x}\right]_{x=\ell} = 0}$$

b) At $x=0$ the string is fixed

B.C.: $u(0,t)=0$, $\left[\frac{\partial u}{\partial x}\right]_{x=\ell} = 0$

$$\frac{\partial^2 u}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = 0 \qquad u(x,t) = X(x)\Theta(t)$$

$$\frac{c^2}{X}\frac{d^2X}{dx^2} = \frac{1}{\Theta}\frac{d^2\Theta}{dt^2} = -\omega^2 \qquad \frac{\partial^2 u}{\partial x^2} = \Theta\frac{d^2X}{dx^2} \quad \frac{\partial^2 u}{\partial t^2} = X\frac{d^2\Theta}{dt^2}$$

$$\frac{d^2\Theta}{dt^2} + \omega^2\Theta = 0 \qquad \frac{d^2X}{dx^2} + \frac{\omega^2}{c^2}X = 0$$

$$\Theta = C_1 e^{\lambda_1 t} \qquad X = C_2 e^{\lambda_2 x}$$

$$\lambda_1^2 + \omega^2 = 0 \qquad \lambda_2^2 + \frac{\omega^2}{c^2} = 0$$

$$\lambda_1^2 = -\omega^2 \qquad \lambda_2^2 = -\frac{\omega^2}{c^2}$$

$$\lambda_1 = \pm i\omega \qquad \lambda_2 = \pm i\frac{\omega}{c}$$

$$\Theta = C_1 e^{\pm i\omega t} \qquad X = C_2 e^{\pm i\frac{\omega x}{c}}$$

$$\Theta = A\cos\omega t + B\sin\omega t \qquad X = C\cos\frac{\omega x}{c} + D\sin\frac{\omega x}{c}$$

B.C. 1 $\quad u(0,t) = X(0)\Theta(t) = 0$

$$X(0) = 0$$

$$X(0) = C = 0$$

$$X = D\sin\frac{\omega x}{c}$$

B.C. 2 $\quad \left[\frac{\partial u}{\partial x}\right]_{x=l} = 0$

$$\frac{\partial u}{\partial x} = \frac{dX}{dx}\Theta(t)$$

$$= D\frac{\omega}{c}\cos\frac{\omega x}{c}\Theta(t)$$

$$\left[\frac{\partial u}{\partial x}\right]_{x=l} = D\frac{\omega}{c}\cos\frac{\omega l}{c}\Theta(t) = 0$$

$$\frac{\omega_n l}{c} = \frac{n\pi}{2}, \quad n = 1, 3, 5, \ldots$$

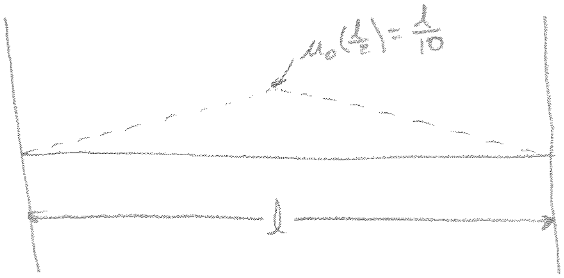
$$\omega_n = \frac{n\pi c}{2l}, \quad n = 1, 3, 5$$

$$u(x,t) = X(x)\Theta(t) \quad \{D = 1\}$$

$$= \sin\frac{n\pi x}{2l}\left[A\cos\frac{n\pi ct}{2l} + B\sin\frac{n\pi ct}{2l}\right], \quad n = 1, 3, 5, \ldots$$

$$\boxed{u(x,t) = A\sin\frac{n\pi x}{2l}\cos\frac{n\pi ct}{2l} + B\sin\frac{n\pi x}{2l}\sin\frac{n\pi ct}{2l}, \quad n = 1, 3, 5, \ldots}$$

8-3



$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} + B_n \sin\frac{n\pi x}{l}\sin\frac{n\pi ct}{l}\right)$$

$$A_n = \frac{2}{l}\int_0^l u_0(x)\sin\frac{n\pi x}{l}\,dx$$

$$B_n = \frac{2}{n\pi c}\int_0^l v_0(x)\sin\frac{n\pi x}{l}\,dx$$

Determine $u_0(x)$

$$\underset{0 \le x \le \frac{l}{2}}{u_0(x)} = \frac{x}{5} \qquad \underset{l/2 \le x < l}{u_0(x)} = \frac{-x}{5} + \frac{l}{5}$$

$$v_0(x) = 0 \longrightarrow B_n = 0$$

$$A_n = \frac{2}{l}\left[\int_0^{l/2}\frac{x}{5}\sin\frac{n\pi x}{l}\,dx - \int_{l/2}^{l}\frac{x}{5}\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l}\frac{l}{5}\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2}{5l}\left[\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx - \int_{l/2}^{l} x\sin\frac{n\pi x}{l}\,dx + l\int_{l/2}^{l}\sin\frac{n\pi x}{l}\,dx\right]$$

$$\int_a^b x\sin kx\,dx = uv\Big|_a^b - \int_a^b v\,du = -\frac{x}{k}\cos kx\Big|_a^b + \frac{1}{k}\int_a^b \cos kx\,dx$$

$$= -\frac{x}{k}\cos kx\Big|_a^b + \frac{1}{k^2}\sin kx\Big|_a^b$$

$u = x \quad dv = \sin x\,dx$

$du = dx \quad v = -\frac{1}{k}\cos kx$

$$A_n = \frac{2}{5l}\left[-\frac{lx}{n\pi}\cos\frac{n\pi x}{l}\Big|_0^{l/2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\Big|_0^{l/2} + \frac{lx}{n\pi}\cos\frac{n\pi x}{l}\Big|_{l/2}^{l} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\Big|_{l/2}^{l} - \frac{l^2}{n\pi}\cos\frac{n\pi x}{l}\Big|_{l/2}^{l}\right]$$

$$= \frac{2}{5l}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + 0 + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - 0 + \frac{l^2}{n\pi}\cos n\pi - \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} - \frac{l^2}{n^2\pi^2}\sin n\pi + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{l^2}{n\pi}\cos n\pi + \frac{l^2}{n\pi}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{5l}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2} - \frac{l^2}{n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n\pi}\cos\frac{n\pi}{2}\right]$$

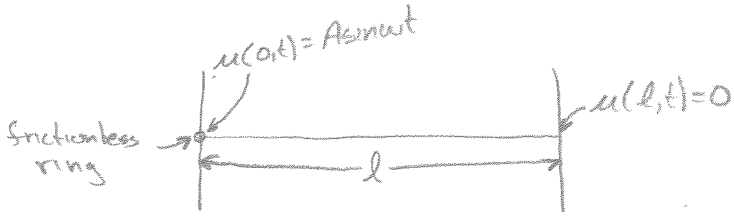
$$= \frac{2}{5l}\left[\frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4l}{5\pi^2}\frac{1}{n^2}\sin\frac{n\pi}{2}$$

$$= \frac{4l}{5\pi^2}\left[1 - \frac{1}{9} + \frac{1}{25} - \cdots\right] \quad \{A_n = 0 \text{ for } n = 2, 4, 6, \ldots\}$$

$$\boxed{u(x,t) = \frac{4\ell}{5\pi^2}\left(\sin\frac{\pi x}{\ell}\cos\frac{\pi c t}{\ell} - \frac{1}{9}\sin\frac{3\pi x}{\ell}\cos\frac{3\pi c t}{\ell} + \frac{1}{25}\sin\frac{5\pi x}{\ell}\cos\frac{5\pi c t}{\ell} - \cdots\right)}$$

8-5



Boundary conditions:

$u(0,t) = A\sin\omega t$

$u(l,t) = 0$

$$u(x,t) = X(x)\,\Theta(t) \qquad X = C\cos\frac{\omega x}{c} + D\sin\frac{\omega x}{c}$$

$$A\sin\omega t = X(0)\,\Theta(t)$$

$$= C\,\Theta(t)$$

$$C = \frac{A\sin\omega t}{\Theta(t)}$$

$$X(l)\,\Theta(t) = 0$$

$$\frac{A\sin\omega t}{\Theta(t)}\cos\frac{\omega l}{c} + D\sin\frac{\omega l}{c} = 0$$

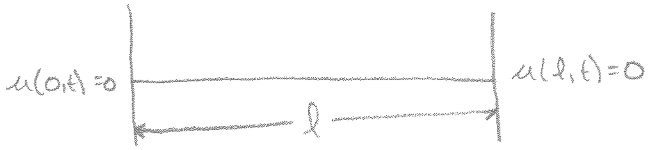
$$D = \frac{-A\sin\omega t}{\Theta(t)}\cot\frac{\omega l}{c}$$

$$u(x,t) = X(x)\,\Theta(t)$$

$$= \left[\frac{A\sin\omega t}{\Theta(t)}\cos\frac{\omega x}{c} - \frac{A\sin\omega t}{\Theta(t)}\cot\frac{\omega l}{c}\sin\frac{\omega x}{c}\right]\Theta(t)$$

$$\boxed{u(x,t) = A\sin\omega t\left[\cos kx - \cot kl\,\sin kx\right], \quad k = \frac{\omega}{c}}$$

8-7



$$F' = b\frac{\partial u}{\partial t}$$

$$\sigma\frac{\partial^2 u}{\partial t^2} = \tau\frac{\partial^2 u}{\partial x^2} + b\frac{\partial u}{\partial t}$$

$$\sigma\frac{\partial^2 u}{\partial t^2} + b\frac{\partial u}{\partial t} = \tau\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} + \frac{b}{\sigma}\frac{\partial u}{\partial t} = \frac{\tau}{\sigma}\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} + \frac{b}{\sigma}\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}$$

$$\frac{\partial^2 u}{\partial t^2} - \frac{b}{\sigma}\frac{\partial u}{\partial t} = -\omega^2 \qquad c^2\frac{\partial^2 u}{\partial x^2} = -\omega^2$$

$$u(x,t) = X(x)\,\Theta(t)$$

$$\frac{\partial^2 u}{\partial t} = X\frac{d^2\Theta}{dt^2} \qquad \frac{\partial^2 u}{\partial x^2} = \Theta\frac{d^2X}{dx^2}$$

$$\frac{\partial u}{\partial t} = X\frac{d\Theta}{dt}$$

$$X\frac{d^2\Theta}{dt^2} + \frac{b}{\sigma}X\frac{d\Theta}{dt} = c^2\Theta\frac{d^2X}{dx^2}$$

$$\frac{1}{\Theta}\frac{d^2\Theta}{dt^2} + \frac{b}{\sigma}\frac{1}{\Theta}\frac{d\Theta}{dt} = \frac{c^2}{X}\frac{d^2X}{dx^2} = -\omega^2$$

$$\frac{d^2\Theta}{dt^2} + \frac{b}{\sigma}\frac{d\Theta}{dt} + \omega^2\Theta = 0$$

$$\Theta = A_1 e^{\lambda_1 t}$$

$$\lambda_1^2 + \frac{b}{\sigma}\lambda_1 + \omega^2 = 0$$

$$\lambda_1 = \frac{-\frac{b}{\sigma} \pm \sqrt{\left(\frac{b}{\sigma}\right)^2 - 4\omega^2}}{2}$$

$$\lambda_1 = \frac{-b}{2\sigma} \pm \sqrt{\left(\frac{b}{2\sigma}\right)^2 - \omega^2}$$

$$\Theta = A_1 e^{\frac{-b}{2\sigma} \pm \sqrt{\left(\frac{b}{2\sigma}\right)^2 - \omega^2}} = A_1 e^{\frac{-b}{2\sigma} \pm i\sqrt{\omega^2 - \left(\frac{b}{2\sigma}\right)^2}}$$

$$\Theta = e^{\frac{-b}{2\sigma}t}\left(A\cos\left\{\left[\omega^2 - \left(\frac{b}{2\sigma}\right)^2\right]^{1/2} t + \theta\right\}\right)$$

$$\frac{d^2X}{dx^2} + \frac{\omega^2}{c^2}X = 0$$

$$X = A_2 e^{\lambda_2 x}$$

$$\lambda_2^2 + \frac{\omega^2}{c^2} = 0$$

$$\lambda_2 = \pm i\frac{\omega}{c}$$

$$X = C\cos\frac{\omega x}{c} + D\sin\frac{\omega x}{c}$$

$$X(0) = C = 0$$

$$X(l) = D\sin\frac{\omega l}{c} = 0$$

$$\frac{\omega l}{c} = n\pi$$

$$\omega_n = \frac{n\pi c}{l}, \quad n = 1, 2, 3, \ldots$$

$X = \sin\frac{n\pi c}{\ell}x \quad \{D=1\}$

$u(x,t) = X(x)\,\Theta(t)$

$$\boxed{u(x,t) = A e^{-\frac{b}{2\sigma}t} \sin\frac{n\pi c x}{\ell} \cos\left\{\left(\frac{n^2\pi^2c^2}{\ell^2} - \frac{b^2}{4\sigma^2}\right)^{1/2} t + \Theta_0\right\},\ n=1,2,3,\ldots}$$

8-9

The vertical force, F_v, due to tension is

$$F_v = \tau \frac{\partial u}{\partial x}$$

Therefore, the frictional force due to the ring is

$$F_f = -F_v$$
$$F_f = -\tau \frac{\partial u}{\partial x}$$

Given that F_f is proportional to its velocity, this gives

$$-\tau \frac{\partial u}{\partial x}\bigg|_{x=0} = b \frac{\partial u}{\partial t}\bigg|_{x=0}$$

as a boundary condition. Now,

$$u(x,t) = f(\xi) + g(\eta) \quad , \quad \xi = x - ct \ , \ \eta = x + ct$$
$$= f(x-ct) + g(x+ct)$$

($f(\xi)$ is a wave moving to the right and $g(\eta)$ is a wave moving to the left)

$$\frac{\partial u}{\partial x} = \frac{df}{d\xi}\frac{\partial \xi}{\partial x} + \frac{dg}{d\eta}\frac{\partial \eta}{\partial x}$$
$$= \frac{df}{d\xi} + \frac{dg}{d\eta}$$

$$-\tau \frac{\partial u}{\partial x}\bigg|_{x=0} = -\tau \frac{df}{d\xi}\bigg|_{x=0} - \tau \frac{dg}{d\eta}\bigg|_{x=0} \quad , \quad d\xi = dx - cdt \ , \ d\eta = dx + cdt$$
$$= -\tau \frac{df}{-cdt} - \tau \frac{dg}{cdt}$$
$$= \frac{\tau}{c}\frac{df}{dt} - \frac{\tau}{c}\frac{dg}{dt}$$

$$\frac{\partial u}{\partial t} = \frac{df}{d\xi}\frac{\partial \xi}{\partial t} + \frac{dg}{d\eta}\frac{\partial \eta}{\partial t} \quad , \quad \frac{\partial \xi}{\partial t} = -c \ , \ \frac{\partial \eta}{\partial t} = c$$
$$= -c\frac{df}{d\xi} + c\frac{dg}{d\eta}$$

$$b\frac{\partial u}{\partial t}\bigg|_{x=0} = -bc\frac{df}{-cdt} + bc\frac{dg}{cdt}$$
$$= b\frac{df}{dt} + b\frac{dg}{dt}$$

B.C. $-\tau \frac{\partial u}{\partial x}\Big|_{x=0} = b \frac{\partial u}{\partial t}\Big|_{x=0}$

$$\frac{\tau}{c}\frac{df}{dt} - \frac{\tau}{c}\frac{dg}{dt} = b\frac{df}{dt} + b\frac{dg}{dt}$$

$$\frac{df}{dt}\left(\frac{\tau}{c} - b\right) = \frac{dg}{dt}\left(\frac{\tau}{c} + b\right)$$

$$\frac{df}{dt}\left(\frac{\tau - bc}{c}\right) = \frac{dg}{dt}\left(\frac{\tau + bc}{c}\right)$$

$$\frac{df}{dt}(\tau - bc) = \frac{dg}{dt}(\tau + bc)$$

$$\frac{dg}{dt} = \left(\frac{\tau - bc}{\tau + bc}\right)\frac{df}{dt}$$

Integrating both sides gives

$$g(\eta) = \left(\frac{\tau - bc}{\tau + bc}\right) f(\xi)$$

where $g(\eta)$ is the reflected wave.

When $bc < \tau$ there is no phase change upon reflection and the reflected amplitude is less than the amplitude of $f(\xi)$. When $bc > \tau$ a $180°$ phase change occurs upon reflection and its amplitude is less than $f(\xi)$.

Limiting cases:

i) $b \ll 1$

$g(\eta) \cong f(\xi)$

no phase change

no change in amplitude

ii) $b \gg 1$

$g(\eta) \cong -f(\xi)$

$180°$ phase change

no change in amplitude

For no reflection

$g(\eta) = 0$

$0 = \left(\frac{\tau - bc}{\tau + bc}\right) f(\xi)$

$0 = \tau - bc$

$b = \frac{\tau}{c} \quad \left(c = \left(\frac{\tau}{\sigma}\right)^{1/2}\right)$

$b = (\sigma\tau)^{1/2}$

8-13

Eq.(8.139)

$$\frac{\partial \vec{V}}{\partial t} + \vec{V}\cdot\vec{\nabla}\vec{V} + \frac{1}{\rho}\vec{\nabla}p = \frac{\vec{f}}{\rho}$$

$$\frac{\partial}{\partial t}\rho\vec{V} + \rho\vec{V}\cdot\vec{\nabla}\vec{V} + \vec{\nabla}p = \vec{f}$$

$$\vec{r}\times\frac{\partial}{\partial t}\rho\vec{V} + \vec{r}\times\rho\vec{V}(\vec{V}\cdot\vec{\nabla}) + \vec{r}\times\vec{\nabla}p = \vec{r}\times\vec{f}$$

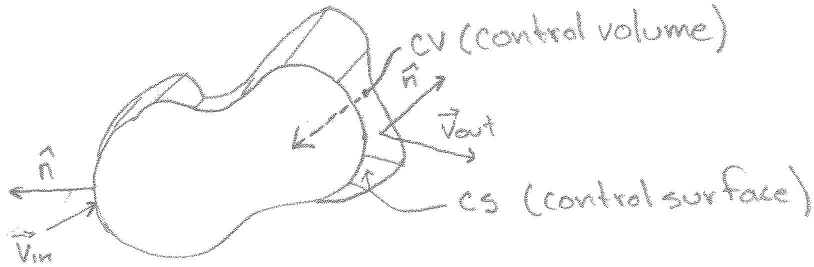
$$\frac{\partial}{\partial t}(\vec{r}\times\rho\vec{V}) + (\vec{V}\cdot\vec{\nabla})\vec{r}\times\rho\vec{V} = \vec{r}\times\vec{f} - \vec{r}\times\vec{\nabla}p$$

Let $\vec{H} = \vec{r}\times\rho\vec{V}$, then

$$\frac{\partial\vec{H}}{\partial t} + \vec{H}(\vec{V}\cdot\vec{\nabla}) = \vec{r}\times\vec{f} - \vec{r}\times\vec{\nabla}p$$
$$= \vec{r}\times(\vec{f} - \vec{\nabla}p)$$

$$\boxed{\frac{\partial\vec{H}}{\partial t} + \vec{H}(\vec{V}\cdot\vec{\nabla}) = Q \quad , \; Q = \vec{r}\times(\vec{f} - \vec{\nabla}p)}$$

8-15



Let B represent any amount (mass, linear momentum, angular momentum, energy, etc.)

$\beta = \frac{dB}{dm}$ be the amount per unit mass

Then

$$B_{cv} = \int_{cv} \beta \rho \, dV$$

The flux of B entering and leaving CS is

$$\int_{cs} \beta \rho \hat{n} \cdot \vec{v} \, dt \, dS \rightarrow B = B_{cv} + B_{flux}$$

The time rate of change of B is

$$\frac{d}{dt} B = \frac{d}{dt}\left(\int_{cv} \beta \rho \, dV\right) + \int_{cs} \beta \rho \, \hat{n} \cdot \vec{v} \, dS$$

Now, B is any amount. So, we use the energy per unit mass

$$\beta = \tfrac{1}{2} v^2 - G + u \quad \{\text{Kinetic, gravitational, internal}\}$$

$$\frac{d}{dt}\left(\int_{cv} \left(\tfrac{1}{2}\rho v^2 - \rho G + \rho u\right) dV\right) + \int_{cs} \hat{n} \cdot \vec{v} \left(\tfrac{1}{2}\rho v^2 - \rho G + \rho u\right) dS = \frac{d}{dt} B_{outside}$$

The outside influence of flux is due to an inward pressure potential $\frac{p}{\rho}$ acting on the control surface. Also, the only outside influence acting within the control volume is G

$$\frac{d}{dt} B_{outside} = \frac{d}{dt}\left(\int_{cv} -\rho G \, dV\right) - \int_{cs} \frac{p}{\rho} \rho \hat{n} \cdot \vec{v} \, dS$$

$$= -\int_{cv} \rho \frac{\partial G}{\partial t} dV - \int_{cs} \hat{n} p \cdot \vec{v} \, dS$$

8-17

The perfect gas law in terms of ρ and M:

$$p = \frac{\rho}{M} RT$$

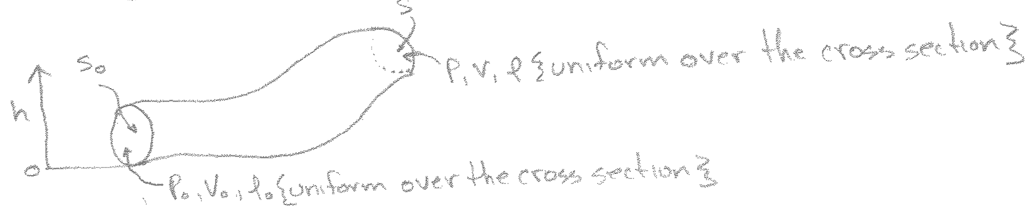
$$u = \int_{\rho_0}^{\rho} \frac{p\,d\rho}{\rho^2} \qquad \rho = \frac{pM}{RT} \rightarrow d\rho = dp\,\frac{M}{RT}$$

$$= \int_{p_0}^{p} p\,dp\,\frac{M}{RT}\left(\frac{RT}{pM}\right)^2$$

$$= \int_{p_0}^{p} \frac{RT}{M}\,\frac{dp}{p}$$

$$\boxed{u = \frac{RT}{M}\ln\left(\frac{p}{p_0}\right)}$$

The flow of the gas is steady and isothermal.



For steady flow

$$\int_S \left(\tfrac{1}{2}\rho v^2 + p - \rho G_0 + \rho u\right) v\,dS = \int_{S_0}\left(\tfrac{1}{2}\rho v_0^2 + p_0 - \rho G + \rho u_0\right) v_0\,dS_0$$

$$G(h) = \int_{h_0=0}^{h} \vec{g}\cdot d\vec{h}$$

$$= -gh \rightarrow G_0 = G(0) = 0$$

$$u_0 = \frac{RT}{M}\ln\left(\frac{p_0}{p_0}\right)$$

$$= 0$$

Equating the pressure terms, p and p_0:

$$pvS = p_0 v_0 S_0$$

$$p = p_0\,\frac{v_0 S_0}{vS}$$

Then

$$u = \frac{RT}{M}\ln\left(\frac{v_0 S_0}{vS}\right)$$

Also, for steady flow

$$\frac{d}{dt}\left\{\left(\frac{1}{2}\rho V^2 + p - \rho G + \rho u\right)\delta V\right\} = 0 \quad \left\{\frac{\partial \rho}{\partial t} = 0 \text{ and } \frac{\partial G}{\partial t} = 0\right\}$$

$$\frac{1}{2}\rho V^2 + p - \rho G + \rho u = \text{const.} \left\{\text{divide both sides by } \delta m = \rho \delta V\right\}$$

$$\frac{1}{2}V^2 + \frac{p}{\rho} - G + u = \text{const.}$$

$$\frac{1}{2}V^2 + \frac{p}{\rho} - G + u = \frac{1}{2}V_0^2 + \frac{p_0}{\rho_0} - G_0 + u_0$$

$$V^2 + \frac{2p}{\rho} - 2G + 2u = V_0^2 + \frac{2p_0}{\rho_0} - 2G_0 + 2u_0$$

$$u_0 = G_0 = 0$$

$$\frac{2p}{\rho} = \frac{2p_0}{\rho_0} \rightarrow \frac{p}{\rho} = \frac{p_0}{\rho_0} \rightarrow \rho = \rho_0 \frac{p}{p_0}$$

$$= \rho_0 \frac{p_0 V_0 S_0}{p_0 V S}$$

$$= \rho_0 \frac{V_0 S_0}{V S}$$

$$G = -gh$$

$$V^2 + 2gh + 2u = V_0^2$$

$$V_0^2 - V^2 - 2u = 2gh$$

$$V_0^2 - V^2 - 2\frac{RT}{M}\ln\left(\frac{V_0 S_0}{VS}\right) = 2gh$$

$$\boxed{\begin{aligned} &V_0^2 - V^2 + 2\frac{RT}{M}\ln\left(\frac{VS}{V_0 S_0}\right) = 2gh \\ &p = p_0 \frac{V_0 S_0}{VS} \\ &\rho = \rho_0 \frac{V_0 S_0}{VS} \end{aligned}}$$

8-19

$$\phi = \frac{a}{r}$$

$$\vec{V} = \vec{\nabla}\phi$$

The gradient in spherical coordinates is

$$\vec{\nabla} = \hat{r}\frac{\partial}{\partial r} + \frac{1}{r}\frac{\partial}{\partial \phi}\hat{\phi} + \frac{1}{r\sin\phi}\frac{\partial}{\partial \theta}\hat{\theta}$$

$$\vec{V} = \hat{r}\frac{\partial}{\partial r}\left(\frac{a}{r}\right)$$

$$\boxed{\vec{V} = \frac{-a}{r^2}\hat{r}}$$

8-21

Show that the normal modes of vibration of a fluid in a box have the velocity oscillation 90° out of phase with the pressure at any point.

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \qquad \frac{\partial v_y}{\partial t} = \frac{-1}{\rho_0}\frac{\partial p'}{\partial y} \qquad \frac{\partial v_z}{\partial t} = \frac{-1}{\rho_0}\frac{\partial p'}{\partial z}$$

$$p' = (A\cos\omega_{\ell mn}t + B\sin\omega_{\ell mn}t)\cos\frac{\ell\pi x}{L_x}\cos\frac{m\pi y}{L_y}\cos\frac{n\pi z}{L_x} \quad (8.225)$$

Since the partial of p' is with respect to the space coordinates and the partial of v is with respect to time, it's clear that after finding $\frac{\partial p'}{\partial x}$, $\frac{\partial p'}{\partial y}$, and $\frac{\partial p'}{\partial z}$ then integrating with respect to time, the time dependent sine and cosine velocity terms will be 90° out of phase with p'.

The reason the velocity is out of phase by 90° with p' is due to Bernoulli's theorem:

$$\frac{v^2}{2} + \frac{p}{\rho} - \vec{G} + u = \text{a constant}$$

Here, G and u are zero. So,

$$v^2 + p = \text{a constant}$$

This states that when the particle velocity is zero the pressure is a maximum.

There is no discrepency with the fact that a plane wave velocity and pressure are in phase. This is because a plane wave propagates with pressure whereas particle velocities oscillate about an equalibrium point and are stationary where pressure is a maximum.

8-23

a) Find the fluid velocity for the wave

$$p' = A\cos\frac{l\pi x}{L_x}\cos\frac{m\pi y}{L_y}\cos(k_z z - \omega t)$$

$$\frac{\partial v_x}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \quad \frac{\partial v_y}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \quad \frac{\partial v_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z}$$

$$-\frac{1}{\rho_0}\frac{\partial p'}{\partial x} = -\frac{1}{\rho_0}\left(-\frac{Al\pi}{L_x}\sin\frac{l\pi x}{L_x}\cos\frac{m\pi y}{L_y}\cos(k_z z - \omega t)\right)$$

$$-\frac{1}{\rho_0}\frac{\partial p'}{\partial y} = -\frac{1}{\rho_0}\left(-\frac{Am\pi}{L_y}\cos\frac{l\pi x}{L_x}\sin\frac{m\pi y}{L_y}\cos(k_z z - \omega t)\right)$$

$$-\frac{1}{\rho_0}\frac{\partial p'}{\partial z} = -\frac{1}{\rho_0}\left(-Ak_z\cos\frac{l\pi x}{L_x}\cos\frac{m\pi y}{L_y}\sin(k_z z - \omega t)\right)$$

$$v_x = \int \frac{Al\pi}{L_x}\sin\frac{l\pi x}{L_x}\cos\frac{m\pi y}{L_y}\cos(k_z z - \omega t)\frac{dt}{\rho_0}$$

$$\boxed{v_x = \frac{-Al\pi}{\omega\rho_0 L_x}\sin\frac{l\pi x}{L_x}\cos\frac{m\pi y}{L_y}\sin(k_z z - \omega t)}$$

$$v_y = \int \frac{Am\pi}{L_y}\cos\frac{l\pi x}{L_x}\sin\frac{m\pi y}{L_y}\cos(k_z z - \omega t)\frac{dt}{\rho_0}$$

$$\boxed{v_y = \frac{-Am\pi}{\omega\rho_0 L_y}\cos\frac{l\pi x}{L_x}\sin\frac{m\pi y}{L_y}\sin(k_z z - \omega t)}$$

$$v_z = \int \frac{Ak_z}{\rho_0}\cos\frac{l\pi x}{L_x}\cos\frac{m\pi y}{L_y}\sin(k_z z - \omega t)$$

$$\boxed{v_z = \frac{Ak_z}{\omega\rho_0}\cos\frac{l\pi x}{L_x}\cos\frac{m\pi y}{L_y}\cos(k_z z - \omega t)}$$

b) Calculate the mean rate of power flow through the pipe.

$$P_{ave} = \langle p'v\rangle_{ave} = \frac{\langle p'^2\rangle_{ave}}{(\rho_0 B)^{1/2}} \qquad B^{1/2} = c\rho_0^{1/2}$$

$$\langle p'^2\rangle_{ave} = \frac{1}{T}\int_0^T p'^2\,dt \qquad T = \frac{2\pi}{\omega}$$

$$= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} p'^2\,dt$$

$$= \frac{\omega}{2\pi}A^2\cos^2\frac{l\pi x}{L_x}\cos^2\frac{m\pi y}{L_y}\int_0^{\frac{2\pi}{\omega}}\cos^2(k_z z - \omega t)\,dt$$

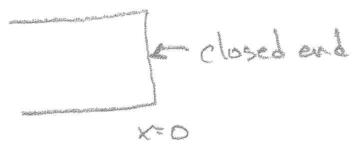
The average value of $\cos^2(k_z z - \omega t)$ is $\frac{1}{2}$, then

$$\boxed{P_{ave} = \frac{\omega}{4\pi(\rho_0 B)^{1/2}} A^2 \cos^2 \frac{l\pi x}{L_x} \cos^2 \frac{m\pi y}{L_y}}$$

8-25

$$p' = A\cos\frac{l\pi x}{L_x}\cos\frac{m\pi y}{L_y}\cos(k_z z-\omega t)$$ {plane wave in +z-direction}

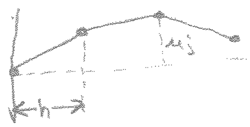
Boundary Condition: $v_z(0)=0$ {this is the boundary condition for a plane wave in a fluid}



The reflected plane wave will simply be the plane wave p' traveling in the opposite direction. Namely

$$\boxed{p_R' = A\cos\frac{l\pi x}{L_x}\cos\frac{m\pi y}{L_y}\cos(k_z z+\omega t)}$$

8-27 Calculate the group velocity of the wave of a discrete string and compare to the phase velocity. Show that the two are very nearly equal for wavelengths much greater than h.



$u_j = A\cos(kx_j - \omega t), \quad x_j = jh$

For a discrete string

$$\cos p = 1 - \frac{\omega^2 mh}{2\tau}, \quad p = kh \quad \{8.73\}$$

$$1 - \cos(kh) = \frac{\omega^2 mh}{2\tau}$$

$$\frac{1-\cos(kh)}{2} = \frac{\omega^2 mh}{4\tau}$$

$$\sin^2\left(\frac{kh}{2}\right) = \frac{\omega^2 mh}{4\tau}$$

$$\sin\left(\frac{kh}{2}\right) = \frac{\omega}{2}\left(\frac{mh}{\tau}\right)^{1/2}$$

$$\omega = \left(\frac{\tau}{mh}\right)^{1/2} 2\sin\left(\frac{kh}{2}\right)$$

$$V_g = \frac{d\omega}{dk}$$

$$= \left(\frac{\tau}{mh}\right)^{1/2} 2\left(\frac{h}{2}\right)\cos\left(\frac{kh}{2}\right)$$

$$\boxed{V_g = \left(\frac{\tau h}{m}\right)^{1/2} \cos\left(\frac{kh}{2}\right)}$$

$$c = \frac{\omega}{k}$$

$$= \left(\frac{\tau}{mh}\right)^{1/2} \frac{2}{k} \sin\left(\frac{kh}{2}\right)$$

$$\boxed{c = \left(\frac{\tau h}{m}\right)^{1/2} \frac{2}{kh} \sin\left(\frac{kh}{2}\right)}$$

$$k = \frac{2\pi}{\lambda}$$

$$v_g = \left(\frac{\tau h}{m}\right)^{1/2} \cos\left(\frac{kh}{2}\right)$$

$$= \left(\frac{\tau h}{m}\right)^{1/2} \cos\left(\frac{\pi h}{\lambda}\right)$$

$$c = \left(\frac{\tau h}{m}\right)^{1/2} \frac{2}{kh} \sin\left(\frac{kh}{2}\right)$$

$$= \left(\frac{\tau h}{m}\right)^{1/2} \frac{\lambda}{\pi h} \sin\left(\frac{\pi h}{\lambda}\right)$$

For $\lambda >> h$

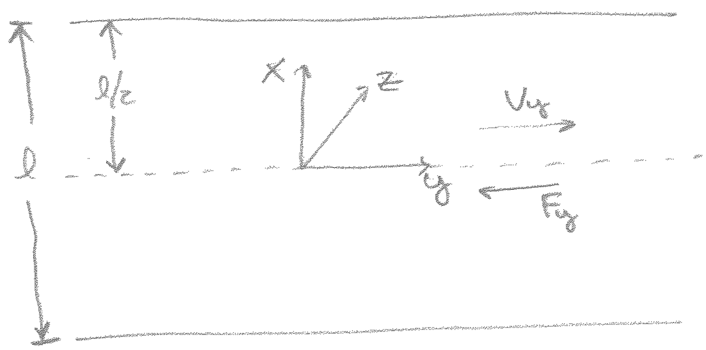
$\cos\left(\frac{\pi h}{\lambda}\right) \to 1$ and $\frac{\lambda}{\pi h} \sin\left(\frac{\pi h}{\lambda}\right) \to \frac{\lambda}{\pi h} \cdot \frac{\pi h}{\lambda} \to 1$

Therefore, when $\lambda >> h$

$$v_g \doteq \left(\frac{\tau h}{m}\right)^{1/2}$$

$$c \doteq \left(\frac{\tau h}{m}\right)^{1/2}$$

8-29 Given η and I' for flow between two infinite parallel plates find the velocity distribution and the pressure gradient parallel to the plates. Assume the pressure varies only in the direction of flow. I' is the current flow per unit length.



$$\eta = \frac{F_y/A_{yz}}{\frac{dV_y}{dx}}$$

$F_y/A_{yz} = \tau$ {shear stress} $\rightarrow \tau = \eta \frac{dV_y}{dx}$

Sum of forces must be zero!

$$F_{press} + F_y = 0$$

$$dp\,dx\,dz + d\tau\,dy\,dz = 0$$

$$-dp\,dx = d\tau\,dy$$

$$-\frac{dp}{dy} = \frac{d\tau}{dx}$$

$$= \frac{d}{dx}\left(\eta \frac{dV_y}{dx}\right)$$

$$-\frac{dp}{dy}\frac{1}{\eta} = \frac{d^2V_y}{dx^2}$$

$$-\frac{1}{\eta}\frac{dp}{dy}dx = \frac{1}{dx}d^2V_y$$

$$-\frac{1}{\eta}\frac{dp}{dy}x + A = \frac{1}{dx}dV_y$$

$$-\frac{1}{\eta}\frac{dp}{dy}x\,dx + A\,dx = dV_y$$

$$-\frac{1}{\eta}\frac{dp}{dy}\frac{x^2}{2} + Ax + B = V_y$$

$v_y = 0$ at $x = \ell/2$ and $-\ell/2$

$$0 = -\frac{1}{\eta}\frac{dp}{dy}\left(\frac{\ell^2}{8}\right) + A\left(\frac{\ell}{2}\right) + B$$

$$B = \frac{1}{\eta}\frac{dp}{dy}\frac{\ell^2}{8} - A\frac{\ell}{2}$$

$$0 = -\frac{1}{\eta}\frac{dp}{dy}\left(\frac{\ell^2}{8}\right) - A\frac{\ell}{2} + B$$

$$0 = -\frac{1}{\eta}\frac{dp}{dy}\frac{\ell^2}{8} - A\frac{\ell}{2} + \frac{1}{\eta}\frac{dp}{dy}\frac{\ell^2}{8} - A\frac{\ell}{2}$$

$$0 = -A\ell$$

$$0 = A$$

$$B = \frac{1}{\eta}\frac{dp}{dy}\frac{\ell^2}{8}$$

$$v_y = -\frac{1}{\eta}\frac{dp}{dy}\frac{x^2}{2} + \frac{1}{\eta}\frac{dp}{dy}\frac{\ell^2}{8}$$

$$v_y = \frac{1}{\eta}\frac{dp}{dy}\left(\frac{\ell^2}{8} - \frac{x^2}{2}\right)$$

$$I = \iint \rho v_y \, dS \qquad dS = dx\,dz$$

$$= \frac{\rho}{\eta}\frac{dp}{dy}\int\int_{-\ell/2}^{\ell/2}\left(\frac{\ell^2}{8} - \frac{x^2}{2}\right)dx\,dz$$

$$= \frac{\rho}{\eta}\frac{dp}{dy}\int \left.\frac{\ell^2}{8}x - \frac{x^3}{6}\right|_{-\ell/2}^{\ell/2} dz$$

$$= \frac{\rho}{\eta}\frac{dp}{dy}\left[\frac{\ell^3}{8} - \frac{\ell^3}{24}\right] z$$

$$\frac{I}{z} = I' = \frac{\rho}{\eta}\frac{dp}{dy}\frac{\ell^3}{12}$$

$$\boxed{\frac{dp}{dy} = \frac{12\eta I'}{\rho \ell^3}}$$

$$v_y = \frac{1}{\eta}\frac{12\eta I'}{\rho\ell^3}\left(\frac{\ell^2}{8} - \frac{x^2}{2}\right)$$

$$\boxed{v_y = \frac{3\eta I'}{2\rho\ell^3}\left(\ell^2 - 4x^2\right)}$$